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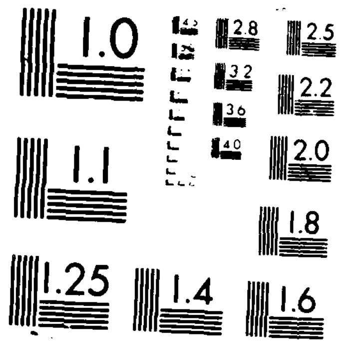

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Let $X=(X_1,\ldots,X_d)$ and $V=(V_1,\ldots,V_d)$ be processes on $[0,1]$ or R_+, such that X is exchangeable while V is predictable. Under suitable conditions on X and V, the expression $E\pi_j \int V_j dX_j$ will only depend on the marginal distributions of X and V. From statements of this type in discrete or continuous time, one may easily derive a variety of old and new results on predictable transformations which preserve the distribution of an exchangeable sequence or process. The same method yields a general result about reduction of continuous local martingales and marked point processes to independent Gaussian and Poisson random fields.

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DECOUPLING IDENTITIES AND PREDICTABLE TRANSFORMATIONS IN EXCHANGEABILITY

by

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DECOUPLING IDENTITIES AND PREDICTABLE TRANSFORMATIONS IN EXCHANGEABILITY

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Abstract: Let $X=(X_1,\dots,X_d)$ and $V=(V_1,\dots,V_d)$ be processes on $[0,1]$ or R_+, such that X is exchangeable while V is predictable. Under suitable conditions on X and V, the expression $E\prod_j \int V_j dX_j$ will only depend on the marginal distributions of X and V. From statements of this type in discrete or continuous time, one may easily derive a variety of old and new results on predictable transformations which preserve the distribution of an exchangeable sequence or process. The same method yields a general result about reduction of continuous local martingales and marked point processes to independent Gaussian and Poisson random fields.

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1. Introduction

A finite or infinite sequence of random variables $\xi=(\xi_1,\xi_2,\dots)$ is said to be exchangeable, if any sequence $(\xi_{k_1},\xi_{k_2},\dots)$ with distinct non-random indices $k_1,k_2,\dots$ has the same distribution. The predictable sampling theorem of [12] (cf. Doob (1936) and [10] for special cases) states that the invariance extends to predictable transformations, in the sense that

$$(\xi_{\tau_1},\xi_{\tau_2},\dots) \overset{d}{=} (\xi_1,\xi_2,\dots), \tag{1}$$

for any sequence of a.s. distinct predictable stopping times $\tau_1,\tau_2,\dots$ taking values in the index set of ξ. Here predictability may be defined in terms of the induced filtration, or more generally (cf. [12]).

The above result can be restated in terms of sums of the form $\sum\xi_k\eta_k$, where $\eta=(\eta_1,\eta_2,\dots)$ is a predictable random sequence, such that the associated counting measure $\mu_\eta(B)=\sum\delta_{\eta_k}(B)=\sum 1_B(\eta_k)$ is a.s. non-random. In fact, it is easily seen to be equivalent (under suitable summability conditions) that the distribution of $\sum\xi_k\eta_k$ should only depend on $P\xi^{-1}$ and μ_η. In particular, we get the same distribution if we take ξ and η to be independent with the same marginal distributions, so any distributional property may be obtained through decoupling.

In this paper we shall show (under suitable integrability conditions) that the d-th moment $E(\sum\xi_k\eta_k)^d$ may be computed by decoupling already under the weaker assumption that the sums $\sum\eta_k^m$ be non-random for all $m\le d$, or for all $m\le d-1$ when the sequences are infinite. In the easy special case of i.i.d. sequences and d=1 or 2, the results reduce to the classical Wald identities. The results for finite sequences seem to be new already for d=1, and yield some surprising conclusions about finite games (e.g. lotteries, card games), which were discussed extensively in [11].

It is interesting to notice that the predictable sampling property in (1) may be easily recovered from the set of all decoupling identities as above with varying d. In fact, taking μ_η to be non-random, and assuming for simplicity that the ξ_k are bounded while $\eta_k=0$ for all but finitely many k, it is clear that $\sum\eta_k^m$

is non-random for all m, so all moments of the sum $\sum \xi_k \eta_k$ can be computed through decoupling. Since the moments determine the law in this case, we obtain the decoupling property of the distribution, which was noted above to be equivalent to (1). In the same way, extensions of the moment formulas in different directions may be used to prove extensions of (1).

With no extra effort, we shall actually obtain multivariate versions of the moment identities described above. Thus $\xi=(\xi_{jk}, j=1,\dots,d, k=1,2,\dots)$ will be assumed to form an exchangeable sequence in R^d, while $\eta=(\eta_{jk})$ will be a corresponding predictable sequence. It will then be seen in Theorems 3.1 and 3.5 that (under suitable integrability conditions) the product moment

$$E \prod_{j=1}^{d} \sum_k \xi_{jk}\eta_{jk} \tag{2}$$

may be computed by decoupling, provided that the sums

$$\sum_k \prod_{j\in J} \eta_{jk}, \qquad J\subset\{1,\dots,d\}, \tag{3}$$

are a.s. non-random. This result may be used to prove a multivariate version of the predictable sampling theorem (Proposition 6.1), stating that the distribution of a sequence ξ in R^d is invariant under possibly different predictable permutations in the d components, provided that ξ has the same property with respect to non-random permutations.

The situation in continuous time is similar, though both statements and proofs rely on stochastic calculus. Recall (cf. [12]) that a process X on $I=[0,1]$ or R_+ is said to be exchangeable, if it starts at 0 and is continuous in probability, and if any set of increments over disjoint intervals of equal length forms an exchangeable sequence. Such a process X is known to have a right-continuous version, which forms a semimartingale with respect to the induced standard (i.e. right-continuous and complete) filtration $\mathcal{F}^X$. The property corresponding to (1) is the fact (cf. [12]) that the process

$$XU^{-1}(t) = \int 1\{U_s \le t\}\, dX_s, \qquad t\in I, \tag{4}$$

($1\{\cdot\}$ denoting the indicator function of the set in brackets) has the same finite-dimensional distributions as X, whenever the process U is predictable and $\bar{I}$-valued

with measure preserving paths (in the sense that $\lambda U^{-1}=\lambda$ a.s., with λ denoting Lebesgue measure on I). Equivalently, assuming V to be predictable and suitably integrable on I, and such that the measure $\mu_V=\lambda V^{-1}$ is a.s. constant, the distribution of the integral $\int V dX$ will only depend on μ_V and PX^{-1}. Under the weaker assumption that $\int V^m d\lambda$ be non-random for $m\le d$ (or for $m\le d-1$ when $I=R_+$), we shall again obtain decoupling identities for the moments $E(\int V dX)^d$, and similarly in the multivariate case (Theorems 4.1 and 5.1), and as before all these identities together may be used to recover the predictable invariance property $XU^{-1} \overset{d}{=} X$ and its multivariate counterpart (Proposition 6.1).

Only for processes on R_+ do we get a simple explicit expression for the moments. It is not clear whether tractable formulas are obtainable in the other three fundamental cases (processes on [0,1] and finite or infinite sequences), for the moments themselves or for suitable linear combinations. Another open question is whether similar results exist for $Ef(\sum \xi_k \eta_k)$ or $Ef(\int V dX)$ with functions f other than polynomials (unless μ_η or μ_V is assumed to be non-random).

When X has continuous paths, hence in particular for the Brownian bridge and motion, it turns out that constancy of the first and second order integrals $\int V_j d\lambda$ and $\int V_i V_j d\lambda$ (in some cases only of the latter) suffices to obtain moment identities of arbitrary order. Thus we get in this case a correspondingly richer class of predictable transformations which preserve the distribution of X (Proposition 6.3).

Our arguments may be extended to the context of general continuous local martingales M and (quasi-leftcontinuous) point processes N, yielding a wide class of predictable transformations mapping M into a Gaussian random field and N into an independent Poisson field, both defined on abstract spaces. From this result (Theorem 6.4) one can read off the classical reductions of M to a Brownian motion and N to a Poisson process through a random time-change (Kunita & Watanabe (1967), Papangelou (1972)), as well as their multivariate extensions (Knight (1970), Meyer (1971); cf. Ikeda & Watanabe (1981)). One may also deduce a related time change result for integrals with respect to p-stable motion, due in the symmetric case

to Rosiński & Woyczyński (1986). A further application is to substochastic translations of random point fields, as considered in Matthes, Kerstan & Mecke (1978). In the present paper we shall only give some simple examples indicating the power of the general result. Further details and extensions will be given elsewhere. We might mention here that an entirely different type of random time change for random point fields in the plane has been considered by Merzbach & Nualart (1986).

Let us next recall some basic facts from exchangeability theory (cf. [6]). In each of the four cases of discrete or continuous, bounded or unbounded time, there is a de Finetti-type theorem exhibiting the general distribution as a unique mixture of ergodic ones. The infinite ergodic sequences are of course the i.i.d. ones, while the finite ergodic sequences are those obtainable (in distribution) through successive drawing without replacement from a finite population.

An R^d-valued process on [0,1] is ergodic exchangeable iff it is distributed as some process of the form

$$X_t = \alpha t + \sigma B_t + \sum_{k=1}^{\infty} \beta_k (1\{\tau_k \le t\} - t), \qquad t \in [0,1], \tag{5}$$

where α and $\beta_1, \beta_2, \ldots$ are d-vectors with $\sum |\beta_k|^2 < \infty$, while σ is a $(d \times d)$-matrix, $\tau_1, \tau_2, \ldots$ are i.i.d. $U(0,1)$ (uniformly distributed on [0,1]), and B is an independent Brownian bridge on R^d. Note that the summation in (5) holds in the sense of a.s. convergence uniformly on [0,1] (cf. [8]). Write ρ for the covariance matrix $\sigma\sigma^T$ (T for transpose) and β for the counting measure $\sum \delta_{\beta_k}$ on $R^d \setminus \{0\}$, and note that PX^{-1} determines and is determined by the triple (α, ρ, β). An ergodic exchangeable process with this distribution is said to be directed by (α, ρ, β).

The situation for R^d-valued processes on R_+ is similar. Here the ergodic exchangeable processes are the so called Lévy processes, which by definition are right-continuous, starting at 0, and with stationary independent increments. The Lévy-Khinchin representation of an integrable Lévy process X in R^d may be written in the form

$$E\exp(iuX_t) = \exp\left\{ itu\gamma - \tfrac{1}{2}tu\rho u^T + t\int(e^{iux}-1-iux)\nu(dx)\right\}, \quad u\in R^d, \tag{6}$$

for some d-vector γ, some covariance matrix ρ, and some <u>Lévy measure</u> ν on $R^d\setminus\{0\}$ satisfying $\int(|x|^2\wedge|x|)\nu(dx)<\infty$. (Here, u should be interpreted as a row vector, and X, γ and x as column vectors.) In this case, PX^{-1} and (γ,ρ,ν) determine each other uniquely, and we shall say that X is <u>directed</u> by the triple (γ,ρ,ν).

For clarity and convenience, we shall delimit ourselves in this paper to the ergodic cases, which cover most of the interesting applications (including empirical processes, as well as sampling sequences and their continuous time approximations). In discrete time, our results can easily be extended to the non-ergodic case, by conditioning on the (permutation) invariant σ-field (cf. Aldous (1985)). In continuous time the same procedure gives formally the right answers, but seems harder to justify. To get around the difficulties with conditioning, one may instead go through the proofs in this paper, to check that everything carries over with obvious changes to the non-ergodic case. We omit the details.

On the other hand, we do consider the case of general filtrations $\mathcal{F}=(\mathcal{F}_n)$ or $(\mathcal{F}_t)$. Recall [10,12] that a sequence $\xi=(\xi_1,\xi_2,\ldots)$ is said to be $\mathcal{F}$-<u>exchangeable</u>, if it is $\mathcal{F}$-adapted, and if the subsequence $\xi_{n+1},\xi_{n+2},\ldots$ is conditionally exchangeable, given $\mathcal{F}_n$, for every $n\in Z_+$. The definition in continuous time is similar. Note that exchangeability implies $\mathcal{F}$-exchangeability in both cases, with $\mathcal{F}$ as the induced (standard) filtration, and that conversely $\mathcal{F}$-exchangeability with an arbitrary $\mathcal{F}$ implies exchangeability in the ordinary sense. Note also that every ergodic $\mathcal{F}$-exchangeable infinite sequence is $\mathcal{F}$-i.i.d, in the sense of being both i.i.d. and $\mathcal{F}$-Markov, and similarly that every ergodic $\mathcal{F}$-exchangeable process on R_+ is $\mathcal{F}$-Lévy.

If a process X on [0,1] is ergodic $\mathcal{F}$-exchangeable with <u>distinct</u> jump vectors β_k, then X itself has an a.s. representation as in (5) with the τ_k i.i.d. U(0,1), and such that each process $1\{\tau_k\leq t\}$ is $\mathcal{F}$-exchangeable with compensator $\log(1-t\wedge\tau_k)$ (cf. [12]). Though this may not be true in general, it is easy to see that a representation as above will always exist on a suitable extension of the basic probability space $(\Omega,\mathcal{O},P)$ with associated standard filtration $\mathcal{F}$. (See Section 5

for a general discussion of extensions.) Since extensions don't affect the definition of stochastic integrals (cf. Theorem 9.26 in Jacod (1979)), we may henceforth assume for the sake of simplicity that any process X as above has an a.s. representation (5) with the stated properties.

Throughout the paper we shall need efficient existence criteria and maximum inequalities for stochastic sums and integrals, to be provided in Section 2. The results may be new and perhaps of some independent interest already for i.i.d. sequences and Lévy processes (Propositions 2.1 and 2.2), where they follow by iterated use of the BDG (Burkholder-Davis-Gundy) inequalities. The corresponding theory for stochastic integrals $\int V dX$ on $[0,1]$ is somewhat harder, but simplifies when $\int V d\lambda$ is non-random, since in that case (and under suitable moment restrictions) $\int V dX = \int U dM$ for some martingale M and predictable process U. This key result (Proposition 2.6) gives a clue to the decoupling property for d=1. A similar identity may be proved in discrete time, but will not be needed in this paper.

General background on exchangeability is provided by Aldous' (1985) lecture notes, supplemented by the author's papers [6], [7] and [10]-[12]. Standard terminology, notation and results from stochastic calculus will usually be taken for granted, and the reader may e.g. consult Dellacherie & Meyer (1975, 1980) or Jacod (1979) for details. In this paper, Lebesgue integrals will often be written without 'dt' or 'dλ'. We shall further write $a \lesssim b$ instead of $a=O(b)$. Finally, L_p-norms are defined with respect to the basic probability P, if nothing else is stated.

2. Stochastic sums and integrals

Here we shall first study predictable summation with respect to i.i.d. sequences $\xi=(\xi_1,\xi_2,\ldots)$, where the underlying filtration $\mathcal{F}=(\mathcal{F}_n)$ is indexed by $Z_+=\{0,1,\ldots\}$. Recall that a sequence $\eta=(\eta_1,\eta_2,\ldots)$ is said to be $\mathcal{F}$-predictable, if η_n is $\mathcal{F}_{n-1}$-measurable for each $n\in N=\{1,2,\ldots\}$.

Proposition 2.1. Let $\mathcal{F}$ be a filtration on Z_+, and let ξ and η be infinite random sequences in R, such that ξ is $\mathcal{F}$-i.i.d. while η is $\mathcal{F}$-predictable. Fix $p\geq 1$, and write $p'=p\wedge 2$ and $p''=p\vee 2$. Then

$$E\sup_n\left|\sum_{k=1}^n \xi_k\eta_k\right|^p \lesssim |E\xi_1|^p\, E\left\{\sum_{k=1}^\infty |\eta_k|\right\}^p + E|\xi_1|^p\, E\left\{\sum_{k=1}^\infty |\eta_k|^{p'}\right\}^{p''/2}. \tag{1}$$

When the bound is finite, the sequence

$$\mu_n = \sum_{k=1}^n \xi_k\eta_k - (E\xi_1)\sum_{k=1}^n \eta_k, \qquad n\in Z_+, \tag{2}$$

converges a.s. and defines an L_p-martingale on $\bar{Z}_+=\{0,1,\ldots;\infty\}$.

Proof. We may assume that the right-hand side of (1) is finite, and write $\xi_k'=\xi_k-E\xi_k$. Since ξ_k' and η_k are integrable and independent for each k, the products $\xi_k'\eta_k$ form a martingale difference sequence. Hence we get by the BDG inequality

$$E\sup_n\left|\sum_{k=1}^n \xi_k\eta_k\right|^p \lesssim E\left\{\sum_{k=1}^\infty |E\xi_k|\,|\eta_k|\right\}^p + E\sup_n\left|\sum_{k=1}^n \xi_k'\eta_k\right|^p$$

$$\lesssim |E\xi_1|^p\, E\left\{\sum_{k=1}^\infty |\eta_k|\right\}^p + E\left\{\sum_{k=1}^\infty \xi_k'^2\eta_k^2\right\}^{p/2}.$$

Iterating the procedure, we get after m steps, with $2^m\in(p,2p]$,

$$E\sup_n\left|\sum_{k=1}^n \xi_k\eta_k\right|^p \lesssim \sum_{r=0}^{m-1}\left|E\xi_1^{(r)}\right|^{p2^{-r}} E\left\{\sum_{k=1}^\infty |\eta_k|^{2^r}\right\}^{p2^{-r}} + E\left\{\sum_{k=1}^\infty \xi_k^{(m)}\eta_k^{2^m}\right\}^{p2^{-m}}, \tag{3}$$

where we define $\xi_k^{(0)}=\xi_k$, and then recursively

$$\xi_k^{(r+1)} = (\xi_k^{(r)} - E\xi_k^{(r)})^2, \qquad r=0,\ldots,m-1. \tag{4}$$

The above argument is justified by the fact that

$$E|\xi_k^{(r)}|^{p2^{-r}} \lesssim E|\xi_k|^p, \qquad r=0,\ldots,m, \tag{5}$$

which follows recursively from (4), if we write for $r=0,\ldots,m-1$

$$E|\xi_k^{(r+1)}|^{p2^{-r-1}} = E|\xi_k^{(r)} - E\xi_k^{(r)}|^{p2^{-r}} \lesssim E|\xi_k^{(r)}|^{p2^{-r}} + |E\xi_k^{(r)}|^{p2^{-r}} \lesssim E|\xi_k^{(r)}|^{p2^{-r}}.$$

Comparing (1) and (2), it is seen that the first terms agree. For the last term in (2), we get by subadditivity and independence

$$E\left\{\sum_{k=1}^{\infty} \xi_k^{(m)} \eta_k^{2^m}\right\}^{p2^{-m}} \le E\sum_{k=1}^{\infty} |\xi_k^{(m)}|^{p2^{-m}} |\eta_k|^p = E|\xi_1^{(m)}|^{p2^{-m}} E\sum_{k=1}^{\infty} |\eta_k|^p. \tag{6}$$

Note also that, by subadditivity,

$$\left\{\sum_{k=1}^{\infty} |\eta_k|^c\right\}^{p/c} \le \left\{\sum_{k=1}^{\infty} |\eta_k|^{p'}\right\}^{p''/2}, \qquad c \ge p'. \tag{7}$$

We now get (1) by combining (3) and (5)-(7). The last assertion follows, since the martingale in (2) is uniformly integrable when the bound in (1) is finite. □

Stochastic integration with respect to Lévy processes was studied extensively in [9], and the following result extends the simple Corollary 4.1 of [9].

Proposition 2.2. Let $\mathcal{F}$ be a standard filtration on R_+, and let X and V be R-valued processes on R_+, such that X is $\mathcal{F}$-Lévy and directed by (γ, σ^2, ν) while V is $\mathcal{F}$-predictable. Fix $p \ge 1$, and write $p' = p \wedge 2$ and $p'' = p \vee 2$. Then

$$E \sup_t \left|\int_0^t V dX\right|^p \lesssim |\gamma|^p E\left(\int |V|\right)^p + \sigma^p E\left(\int V^2\right)^{p/2}$$
$$+ \left\{\left(\int |x|^{p'} \nu(dx)\right)^{p''/2} + \int |x|^p \nu(dx)\right\} E\left[\left(\int |V|^{p'}\right)^{p''/2} + \int |V|^p\right], \tag{8}$$

in the sense that $\int V dX$ exists and satisfies (8) when the bound is finite. In this case, the process

$$M_t = \int_0^{t+} V dX - \gamma \int_0^t V, \qquad t \ge 0, \tag{9}$$

converges a.s. as $t \to \infty$ and defines an L_p-martingale on $\bar{R}_+ = [0, \infty]$.

For our current and future needs, let us record the well-known norm interpolation formula

$$\|f\|_q \le \|f\|_p \vee \|f\|_r, \qquad 0 < p \le q \le r, \tag{10}$$

valid in arbitrary measure spaces. For a simple proof, note that $\log\|f\|_{1/t}$ is convex in $t > 0$ by Hölder's inequality.

Proof. To prove (8), it is clearly enough to decompose X into its drift, diffusion and purely discontinuous components, and to prove (8) for each. Now (8) is trivial if X is linear, and if X is a Brownian motion the integral process $\int VdX$ is a continuous local martingale with quadratic variation $\int V^2$, so (8) follows by the BDG inequality. It thus remains to consider the case when X is purely discontinuous and centered.

Letting $m \in \mathbb{N}$ with $2^m \in (p, 2p]$, and proceeding by iterated formal application of the BDG inequality, we then get as in the last proof

$$E \sup_t \left| \int_0^{t+} VdX \right|^p \lesssim \sum_{r=1}^{m-1} \left\{ E\int_0^1 (dX)^{2^r} \right\}^{p2^{-r}} E\left\{ \int_0^\infty V^{2^r} \right\}^{p2^{-r}} + E\left\{ \int_0^\infty V^{2^m} (dX)^{2^m} \right\}^{p2^{-m}}, \tag{11}$$

where

$$\int_0^{t+} (dX)^{2^r} = \sum_{s \le t} (\Delta X_s)^{2^r}, \qquad t \ge 0, \quad r \in \mathbb{N}. \tag{12}$$

To justify (11), we need to show that the stochastic integral processes $\int VdX$ and

$$\int_0^{t+} V^{2^r} \left\{ (dX)^{2^r} - E(dX)^{2^r} \right\}, \qquad t \ge 0, \quad r = 1, \ldots, m-1, \tag{13}$$

exist and are local martingales. But this holds by Definition 2.46 in Jacod (1979), provided that the right-hand side of (11) is finite. It is thus enough to show that the latter expression is bounded by the one in (8).

To see this, note that

$$E \sum_{s \le t} |\Delta X_s|^q = t \int |x|^q \nu(dx), \qquad q > 0, \ t \ge 0, \tag{14}$$

and that the jump process J_q on the left of (14) is again $\mathcal{F}$-Lévy, provided that $q \ge p'$. Thus J_q is compensated, for $q \in [p', p]$, by the function on the right of (14), and we get by the subadditivity of $x^{p2^{-m}}$ and the predictability of $|V|^p$

$$E\left\{ \int_0^\infty V^{2^m} |dX|^{2^m} \right\}^{p2^{-m}} \le E \sum_{t>0} |V_t \Delta X_t|^p = E\int_0^\infty |V|^p \int |x|^p \nu(dx). \tag{15}$$

The estimate in (8) now follows from (11) by means of (10), (12), (14) and (15). The last assertion follows from the fact that finiteness in (8) implies uniform integrability of the process $\int VdX$. □

The remainder of this section is devoted to stochastic integration with respect to ergodic exchangeable processes on [0,1], and we begin with a general existence theorem.

Proposition 2.3. *Let $\mathcal{F}$ be a standard filtration on* [0,1], *and let* X *and* V *be* R-*valued processes on* [0,1], *such that* X *is ergodic $\mathcal{F}$-exchangeable and directed by* (α,σ^2,β) *while* V *is $\mathcal{F}$-predictable. Fix* $p\in(0,2]$ *and* $\varepsilon\geq 0$ *with* $\varepsilon>0$ *if* $p>1$, *and such that* $\sum|\beta_k|^p<\infty$, *that* $\sigma=0$ *if* $p<2$, *and that* $\alpha=\sum\beta_k$ *if* $p<1$. *Then* $\int VdX$ *exists on* [0,1] *provided that*

$$\int_0^1|V_t|^p(1-t)^{-\varepsilon}dt<\infty \quad \text{a.s.} \tag{16}$$

Note in particular that (16) holds if $\int|V|^r<\infty$ a.s. for some $r>p$. Weaker conditions for integrability on intervals $[0,t]$ with $t<1$ may be obtained by adaption of the methods in [9].

Here and below we shall use the fact (cf. [12]) that X is a special semimartingale on [0,1] with canonical decomposition of the form

$$X_t = M_t - \int_0^t \frac{X_s-\alpha}{1-s}ds, \qquad t\in[0,1], \tag{17}$$

where M is an L_2-martingale with associated quadratic variation process

$$[M,M]_t = [X,X]_t = \sigma^2 t + \sum_{k=1}^{\infty}\beta_k^2\, 1\{\tau_k\leq t\}, \qquad t\in[0,1]. \tag{18}$$

Proof. When $p\leq 1$, we may clearly assume that $\sigma=0$, $\alpha=\sum\beta_j$, $\sum|\beta_j|^p<\infty$ and $\int|V|^p<\infty$ a.s., and it is then enough to show that $\int VdX$ exists as a Lebesgue-Stieltjes integral, i.e. that

$$\int_0^1|V|\,|dX| = \sum_{j=1}^{\infty}\left|\beta_j V_{\tau_j}\right|<\infty \quad \text{a.s.} \tag{19}$$

To see this, write

$$Y_t = \sum_{j=1}^{\infty}|\beta_j|^p 1\{\tau_j\leq t\}, \qquad t\in[0,1], \tag{20}$$

and note that Y is compensated by the process with density

$$N_t = \frac{1}{1-t}\sum_{k=1}^{\infty}|\beta_k|^p\ 1\{\tau_k>t\}, \qquad t\in[0,1). \tag{21}$$

By subadditivity and dual predictable projection, we get

$$E\left\{\int_0^1 |V|\,|dX|\right\}^p \le E\sum_{j=1}^{\infty}\left|\beta_j V_{\tau_j}\right|^p = E\int_0^1 |V|^p dY = E\int_0^1 |V|^p N. \tag{22}$$

The same relation holds with V replaced by the predictable processes

$$V_n(t) = V(t)\cdot 1\{t\le\sigma_n\}, \qquad t\in[0,1],\ n\in N, \tag{23}$$

where $\sigma_1, \sigma_2, \ldots$ denote the $[0,1]$-valued stopping times

$$\sigma_n = \sup\left\{t\le 1;\ \int_0^t |V|^p N \le n\right\}, \qquad n\in N, \tag{24}$$

so we get

$$\int_0^{\sigma_n+} |V|\,|dX| < \infty \quad \text{a.s.}, \qquad n\in N. \tag{25}$$

To obtain (19), it remains to notice that $\sigma_n=1$ for all sufficiently large n, which holds since N is a positive martingale and therefore a.s. bounded.

Let us next assume that $p\in(1,2]$ and $\sum|\beta_j|^p<\infty$, and that (16) holds for some $\varepsilon>0$. Then $\int|V|<\infty$ a.s., so we may further take $\alpha=0$. Starting with the case when $\sigma=0$, and defining M, Y and N by (17), (20) and (21), we get by (18), Jensen's inequality, subadditivity and dual predictable projection

$$\begin{aligned}\left\{E\left\{\int_0^1 V^2 d[M,M]\right\}^{1/2}\right\}^p &= \left\{E\left\{\sum_j \beta_j^2 V_{\tau_j}^2\right\}^{1/2}\right\}^p \le E\left\{\sum_j \beta_j^2 V_{\tau_j}^2\right\}^{p/2}\\ &\le E\sum_j \left|\beta_j V_{\tau_j}\right|^p = E\int_0^1 |V|^p dY = E\int_0^1 |V|^p N.\end{aligned} \tag{26}$$

Replacing V by the processes V_n in (23), we get from (26)

$$E\left\{\int_0^{\sigma_n+} V^2 d[M,M]\right\}^{1/2} < \infty, \qquad n\in N, \tag{27}$$

with the σ_n given by (24). As before, $\sigma_n=1$ for sufficiently large n, so $\int V dM$ exists by Definition 2.46 in Jacod (1979). If instead p=2 and $\sigma>0$, we get in place of (26)

$$\left\{E\left\{\int_0^1 V^2 d[M,M]\right\}^{1/2}\right\}^2 = E\left\{\sigma^2\int V^2 + \sum_j \beta_j^2 V_{\tau_j}^2\right\} \le E\int_0^1 V_t^2(\sigma^2+N_t)\,dt, \tag{28}$$

and the existence of $\int V dM$ follows as before.

To complete the proof for $p>1$, it remains to show that V is Lebesgue-Stieltjes

integrable with respect to the second component in (17), i.e. that $V_tX_t/(1-t)$ is Lebesgue integrable over $[0,1)$. To see this, conclude from Hölder's inequality that

$$\int_0^1 \frac{|V_tX_t|}{1-t}\,dt \le \left\{\int_0^1 |V_t|^p (1-t)^{-\varepsilon} dt\right\}^{1/p} \left\{\int_0^1 |X_t|^q (1-t)^{-q'} dt\right\}^{1/q}, \tag{29}$$

where $p^{-1}+q^{-1}=1$ and $q'=(1-\varepsilon p^{-1})q<q$, and note that the first factor on the right is a.s. finite by (16). To show that even the second factor is finite a.s., we may assume that $1<q'<q$, since $(1-t)^{-q'}$ is non-decreasing in q'. In that case there is a $p'>p$ satisfying $p'^{-1}+q'^{-1}=1$, and by Theorem 2.1 in [7] we get $|X_t|^{p'} \lesssim 1-t$ a.s. as $t\to 1$, so

$$|X_t|^q(1-t)^{-q'} \lesssim (1-t)^{-q'+q/p'} \quad \text{a.s.},$$

which is integrable over $[0,1)$, since $-q'+q/p' > -q'+q'/p' = -1$. □

For the remainder of this section, we assume that $\mathcal{F}$, X, V, (α,σ^2,β), p and ε are such as in Proposition 2.3. We shall further assume that (1.5) holds for some B and $\tau_1,\tau_2,\ldots$ with the stated properties. Let us write $\bar V=\int V$ when the integral exists.

Proposition 2.4. For $p\le 1$, we have

$$\int_0^1 V\,dX = \sum_{j=1}^{\infty} \beta_j V_{\tau_j} \quad \text{a.s.}, \tag{30}$$

while for $p\ge 1$,

$$\int_0^1 V\,dX = \alpha \bar V + \sigma\int_0^1 V\,dB + \sum_{j=1}^{\infty} \beta_j (V_{\tau_j} - \bar V) \quad \text{a.s.}, \tag{31}$$

where the series on the right converges in probability.

The proof requires a lemma of some independent interest.

Lemma 2.5. For fixed $r\in[0, p^{-1}\wedge 1)$ we have

$$\lim_{n\to\infty} \sup_t \, (t(1-t))^{-r}\left|\sum_{j=n}^{\infty} \beta_j (1\{\tau_j\le t\}-t)\right| = 0 \quad \text{a.s.} \tag{32}$$

If instead $1\le r<p^{-1}$, we get

$$\lim_{n\to\infty} \sup_t \, t^{-r}\left|\sum_{j=n}^{\infty} \beta_j 1\{\tau_j\le t\}\right| = 0 \quad \text{a.s.} \tag{33}$$

The second part is stated here only for completeness and will not be needed in this paper. It follows easily by combination of Theorem 2.1 in [7] and Theorem 3 in [8]. Alternatively, it may be obtained by adaption of the following argument to the case $r \geq 1$.

Proof for $r<1$. We may clearly assume that $p>1$. Let us introduce the martingale $M'_t = X_t/(1-t)$. By the BDG inequality, the exchangeability of $[X,X]$, and the subadditivity of $x^{p/2}$, we get for $0 \leq t \leq t+h \leq 1/2$

$$E[|M'_{t+h}-M'_t|^p|\mathcal{F}_t] \lesssim E[([M',M']_t^{t+h})^{p/2}|\mathcal{F}_t] \lesssim E[([X,X]_t^{t+h})^{p/2}|\mathcal{F}_t]$$

$$\leq E([X,X]_{2h})^{p/2} = E(\sum_j \beta_j^2 1\{\tau_j \leq 2h\})^{p/2}$$

$$\leq E\sum_j |\beta_j|^p 1\{\tau_j \leq 2h\} = 2h \sum_j |\beta_j|^p.$$

Hence we obtain for any $0=t_0<t_1<\ldots<t_n=t\leq 1/2$

$$\sum_k E[|M'_{t_k}-M'_{t_{k-1}}|^p|\mathcal{F}_{t_{k-1}}] \lesssim t\sum_j |\beta_j|^p.$$

We may then conclude from Lemma 2.3 in [9] that, for some constant $c>0$ and for any increasing and continuous function $g: R_+ \rightarrow R_+$

$$P\Big\{\sup_{t\leq 1/2} |M'_t|^p/g(ct\sum_j |\beta_j|^p) \geq \varepsilon\Big\} \leq \frac{2}{\varepsilon}\int_0^\infty \frac{du}{g(u)}, \quad \varepsilon>0. \tag{34}$$

Here the left-hand side depends only on $g(x)$ for $2x \leq c\sum|\beta_j|^p$, so we may choose $g(x)=(x/c)^{pr}$ for such x and let $1/g$ be integrable on $(0,\infty)$, to obtain

$$P\Big\{\sup_{t\leq 1/2} |X_t| t^{-r} \geq \varepsilon\Big\} \lesssim \varepsilon^{-p}\sum_{j=1}^\infty |\beta_j|^p, \quad \varepsilon>0. \tag{35}$$

Applying (35) and the corresponding inequality for $t\geq 1/2$ to the processes in (32), we obtain as $n\rightarrow\infty$

$$\sup_t \Big|(t(1-t))^{-r}\sum_{j=n}^\infty \beta_j(1\{\tau_j\leq t\}-t)\Big| \xrightarrow{P} 0. \tag{36}$$

It follows in particular that the processes on the left have paths in $D[0,1]$. Since the individual terms are independent, it follows by Theorem 3 in [8] that the convergence in (36) is in fact a.s. □

Proof of Proposition 2.4. The result for $p\le 1$ was established in the proof of Proposition 2.3, so it remains to take $p>1$. Since (31) is trivial when the sum in (1.5) is finite, it is enough to prove that $\int V dX_n \xrightarrow{P} 0$, where X_n is the sum in (1.5) for $j\ge n$. Writing M_n and N_n for the associated martingales M and N in (16) and (21), and introducing the stopping times

$$\sigma_n = \sup\left\{t\le 1;\ \int_0^t |V|^p N_n \le 1\right\}, \qquad n\in\mathbb{N}, \tag{37}$$

we get as before, by the BDG inequality and dual predictable projection,

$$E\left|\int_0^{\sigma_n+} V dM_n\right|^p \lesssim E\left\{\int_0^{\sigma_n+} V^2 d[M_n,M_n]\right\}^{p/2} \le E\left\{1\wedge\int_0^1 |V|^p N_n\right\} \le 1. \tag{38}$$

Now $N_n \downarrow 0$ as $n\to\infty$, so $\int |V|^p N_n \to 0$ a.s. by dominated convergence, and therefore $\sigma_n=1$ for all sufficiently large n, while the integral on the left of (38) tends to zero in probability. Thus $\int V dM_n \xrightarrow{P} 0$.

To prove the corresponding result for the compensating term, let $p^{-1}+q^{-1}=1$, $q'=q(1-\varepsilon/p)$, and $r+q'^{-1}=1$, where we may assume that $1\le q'<q$, so that $0\le r<p^{-1}<1$. Using Hölder's inequality as in (29), we get

$$\left|\int_0^1 \frac{V(t)X_n(t)}{1-t}\right| \le \left\{\int_0^1 \frac{|V_t|^p}{(1-t)}\right\}^{1/p}\left\{\int_0^1 (1-t)^{qr-q'}\right\}^{1/q} \sup_t \frac{|X_n(t)|}{(1-t)^r}. \tag{39}$$

Here the first two factors on the right are a.s. finite as before, while the last one tends to zero a.s., by Lemma 2.5. □

The next result gives the fundamental connection to martingales when $\bar V$ is constant, and will play a key role in Section 4. Recall that M is the martingale in (16).

Proposition 2.6. Let $p>1$, and assume that V is a.s. non-random. Then

$$\int_0^1 V_t dX_t = \alpha\bar V + \int_0^1 \left(V_t - \frac{1}{1-t}\int_t^1 V_s ds\right) dM_t \quad \text{a.s.} \tag{40}$$

For the proof we shall need the first part of the following lemma. The second part will be needed later.

Lemma 2.7. Let the function $f: R_+ \to R$ be locally integrable, and define

$$g_t = \frac{1}{t}\int_0^t f_s ds, \quad t > 0.$$

Then we have for any $p > 1$ and $r \geq 0$

$$\int_0^\infty |g_t|^p t^{-r} dt \leq \left(\frac{p}{r+p-1}\right)^p \int_0^\infty |f_t|^p t^{-r} dt. \tag{41}$$

If f is square integrable on $[0,1]$, we have in addition

$$\int_0^1 (f_t - g_t)^2 dt = \int_0^1 (f_t - g_1)^2 dt. \tag{42}$$

Proof. To prove (41), we may clearly assume that $f \geq 0$ and $\int f > 0$, and by monotone convergence we may further assume the support of f to be compact in $(0,\infty)$, so that the left-hand side of (41) is finite and strictly positive. Writing $F_t = tg_t$ and letting $p^{-1}+q^{-1}=1$, we get by partial integration and Hölder's inequality

$$\int_0^\infty g_t^p t^{-r} dt = \int_0^\infty F_t^p t^{-r-p} dt = \frac{p}{r+p-1}\int_0^\infty F_t^{p-1} f_t t^{-r-p+1} dt = \frac{p}{r+p-1}\int_0^\infty g_t^{p-1} f_t t^{-r} dt$$

$$\leq \frac{p}{r+p-1}\left\{\int_0^\infty g_t^p t^{-r} dt\right\}^{1/q}\left\{\int_0^\infty f_t^p t^{-r} dt\right\}^{1/p},$$

from which (41) follows if we divide by the second factor on the right.

In particular, $f \in L_2[0,1]$ implies $g \in L_2[0,1]$, and in that case we get by repeated use of Fubini's theorem

$$\int_0^1 g_s^2 ds = \int_0^1 s^{-2} F_s^2 ds = 2\int_0^1 s^{-2} ds \int_0^s f_t F_t dt = 2\int_0^1 f_t F_t dt \int_t^1 s^{-2} ds$$

$$= 2\int_0^1 f_t F_t (t^{-1}-1) dt = 2\int_0^1 f_t g_t dt - g_1^2.$$

Thus

$$\int (f-g)^2 = \int f^2 - 2\int fg + \int g^2 = \int f^2 - 2\int fg + 2\int fg - g_1^2 = \int (f-g_1)^2. \qquad \square$$

Proof of Proposition 2.6. First note that the stochastic integral in (40) exists by Lemma 2.7 and by the proof of Proposition 2.3. Since M is clearly independent of α, we may assume that $\alpha=0$. Define $M_t' = X_t/(1-t)$ as before, and conclude from Itô's formula and (17) that

$$dX_t = (1-t)dM_t' - M_t' dt = dM_t - M_t' dt. \tag{43}$$

Integrating (stochastically) by parts and using the constancy of $\bar{V}$, we get for $t<1$

$$\int_0^t V_s M'_s ds = M'_t \int_0^t V_s ds - \int_0^{t+} dM'_s \int_0^s V_r dr = \int_0^{t+} dM'_s \int_s^1 V_r dr - M'_t \int_t^1 V_s ds,$$

so by (43)

$$\int_0^{t+} V_s dX_s = \int_0^{t+} (V_s - \frac{1}{1-s} \int_s^1 V_r dr) dM_s + \frac{X_t}{1-t} \int_t^1 V_s ds. \tag{44}$$

Thus (40) follows by dominated convergence for stochastic integrals, provided we can show that the last term in (44) tends to zero as $t \to 1$. To see this, use Hölder's inequality with $p^{-1}+q^{-1}=1$ along with formula (18) above, to obtain

$$\left| \frac{X_t}{1-t} \int_t^1 V \right| \le \frac{|X_t|}{1-t} \left\{ \int_t^1 (1-s)^{\varepsilon q/p} \right\}^{1/q} \left\{ \int_t^1 |V_s|^p (1-s)^{-\varepsilon} \right\}^{1/p} \lesssim |X_t| (1-t)^{-(1-\varepsilon)/p},$$

and recall that the right-hand side goes to zero a.s., by Theorem 2.1 in [7]. □

We conclude this section by proving a maximum inequality, similar to those in Propositions 2.1 and 2.2, for the stochastic integral in Proposition 2.6. Let us then denote the integrand by U, i.e.

$$U_t = V_t - \frac{1}{1-t} \int_t^1 V_s ds, \qquad t \in [0,1). \tag{45}$$

The constant p may now be different from that in Proposition 2.3.

Proposition 2.8. *Assume that $\bar{V}$ exists and is a.s. non-random. Fix constants $p \ge 1$ and $q > 2p$, and write $p' = p \wedge 2$ and $p'' = p \vee 2$. Then*

$$E \sup_t \left| \int_0^t U dM \right|^p \lesssim \sigma^p E \left\{ \int_0^1 V^2 \right\}^{p/2} + \left\{ \sum_j |\beta_j|^{p'} \right\}^{p''/2} \left\{ E \int_0^1 |V|^q \right\}^{p/q}, \tag{46}$$

where the process $\int U dM$ exists as a martingale whenever the right-hand side is finite. In that case also, the series in (31) converges in L_p.

Proof. Assume that the right-hand side of (46) is finite. Then we know that $\int U dM$ exists as a local martingale. To prove (46), conclude from the BDG inequality and (18) that

$$E \sup_t \left| \int_0^t U dM \right|^p \lesssim E \left\{ \int_0^1 U^2 d[M,M] \right\}^{p/2}$$

$$= E \left\{ \sigma^2 \int_0^1 U^2 + \sum_j \beta_j^2 U_{\tau_j}^2 \right\}^{p/2} \lesssim \sigma^p E \left\{ \int_0^1 U^2 \right\}^{p/2} + E \left\{ \sum_j \beta_j^2 U_{\tau_j}^2 \right\}^{p/2}.$$

Since $\int U^2 \lesssim \int V^2$ by Lemma 2.7, it remains to estimate the second term on the right.

If $p \le 2$, we get by subadditivity

$$E\left\{\sum_j \beta_j^2 U_{\tau_j}^2\right\}^{p/2} \le E\sum_j \left|\beta_j U_{\tau_j}\right|^p \le \sum_j |\beta_j|^p \sup_j E\left|U_{\tau_j}\right|^p.$$

If instead $p \ge 2$, we obtain by Hölder's inequality

$$E\left\{\sum_j \beta_j^2 U_{\tau_j}^2\right\}^{p/2} \le \left\{\sum_j \beta_j^2\right\}^{p/2-1} E\sum_j \beta_j^2 \left|U_{\tau_j}\right|^p \le \left\{\sum_j \beta_j^2\right\}^{p/2} \sup_j E\left|U_{\tau_j}\right|^p.$$

Thus it suffices in both cases to show that

$$\sup_j E\left|U_{\tau_j}\right|^p \lesssim \left\{E\int |V|^q\right\}^{p/q}. \tag{47}$$

To see this, use dual predictable projection, Hölder's inequality with $r=(1-p/q)^{-1}$, Fubini's theorem, and Lemma 2.7, to obtain

$$\begin{aligned} E\left|U_{\tau_j}\right|^p = E\int_0^1 |U_t|^p d(1\{\tau_j \le t\}) \quad E\int_0^{\tau} \frac{|U_t|^p}{1-t}\,dt &\le \left\{E\int_0^1 |U|^q\right\}^{p/q}\left\{E\int_0^{\tau}(1-t)^{-r}\right\}^{1/r} \\ &\lesssim \left\{E\int_0^1 |V|^q\right\}^{p/q}\left\{\int_0^1 (1-t)^{1-r}\right\}^{1/r}. \end{aligned}$$

This completes the proof of (46). In particular, it follows by uniform integrability that the process $\int U dM$ is a martingale. To prove the last assertion, write X_n for the sum in (1.5) over indices $j \ge n$, let M_n be the corresponding martingale M in (17), and conclude from Propositions 2.4, 2.6 and 2.8 that

$$E\left|\sum_{j=n}^{\infty} \beta_j (V_{\tau_j} - \bar{V})\right|^p = E\left|\int_0^1 V dX_n\right|^p = E\left|\int_0^1 U dM_n\right|^p \lesssim \left\{\sum_{j=n}^{\infty} |\beta_j|^{p'}\right\}^{p''/2}\left\{E\int_0^1 |V|^q\right\}^{p/q},$$

where the right-hand side tends to zero as $n \to \infty$. $\square$

3. Finite and infinite sequences

Our aim in this section is to prove decoupling identities for exchangeable sequences ξ in R^d. Thus we take the k-th element in ξ to be a random vector $\xi_{.k}=(\xi_{1k},\ldots,\xi_{dk})$, and ξ itself to be an array $\xi=(\xi_{jk})$, where $j\in\{1,\ldots,d\}$ while $k\in N=\{1,2,\ldots\}$ or $k\in\{1,\ldots,n\}$ for some $n\in N$. The associated filtration $\mathcal{F}=(\mathcal{F}_k)$ is then indexed by $Z_+=\{0,1,\ldots\}$ or $\{0,\ldots,n\}$, respectively. We emphasize that $\mathcal{F}$-exchangeability is to be understood in the joint sense in this section, i.e. the random vectors $\xi_{.1},\xi_{.2},\ldots$ are assumed to form an exchangeable sequence. Along with ξ we also consider a predictable sequence $\eta=(\eta_{.k})=(\eta_{jk})$ in R^d, where j and k range over the same index sets as before.

We shall first consider the case of finite sequences ξ and η, both of length $n\in N$. Let us then introduce the sums

$$R_J=\sum_{k=1}^{n}\prod_{j\in J}\xi_{jk},\qquad S_J=\sum_{k=1}^{n}\prod_{j\in J}\eta_{jk},\qquad \emptyset\neq J\subset\{1,\ldots,d\}. \tag{1}$$

Recall that a finite exchangeable sequence $\xi=(\xi_{jk})$ is ergodic, if the counting measure

$$\mu_\xi=\sum_{k=1}^{n}\delta_{\xi_{.k}};\qquad \mu_\xi(B)=\#\{k;\ \xi_{.k}\in B\},\qquad B\subset R^d, \tag{2}$$

is non-random. In this case each R_J is constant, like any function of ξ which is invariant under permutations in index k. Our basic assumption is that even the sums S_J be non-random. In addition to this we shall need a technical condition, to ensure the existence of moments:

(C_1): There exist constants $p_1,\ldots,p_d\geq 1$ with $\sum p_j^{-1}\leq 1$, such that

$$E|\eta_{jk}|^{p_j}<\infty,\qquad j=1,\ldots,d,\quad k=1,\ldots,n.$$

Theorem 3.1. *Let $\mathcal{F}$ be a filtration on $\{0,\ldots,n\}$ and let ξ and η be random n-sequences in R^d, such that ξ is ergodic $\mathcal{F}$-exchangeable while η is $\mathcal{F}$-predictable. Assume that (C_1) is fulfilled, and that S_J is a.s. non-random for every $J\subset\{1,\ldots,d\}$. Then*

$$E_d=E\prod_{j=1}^{d}\sum_{k=1}^{n}\xi_{jk}\eta_{jk}=P_{nd}\{R_J,S_J\}, \tag{3}$$

for some polynomial P_{nd} *in the sums* R_J *and* S_J. *In particular* $E_1=n^{-1}R_1S_1$, *and under the further assumption* $R_j=S_j=0$ *we have*

$$E_2 = \frac{1}{n-1} R_{12}S_{12}, \quad E_3 = \frac{n}{(n-1)(n-2)} R_{123}S_{123}. \tag{4}$$

A slightly more general statement will be proved in Lemma 3.4. But first we need to prove a couple of preliminary results, where the first one will also be useful later on.

Lemma 3.2. For $x_{jk}\in\mathbb{R}$, $j=1,\ldots,d$, $k\in\mathbb{N}$, *with*

$$\prod_{j=1}^{d} \sum_{k=1}^{\infty} |x_{jk}| < \infty, \tag{5}$$

define

$$P = \sum_{(k_j)} \prod_{j=1}^{d} x_{j,k_j}, \tag{6}$$

where the summation extends over all choices of distinct $k_1,\ldots,k_d\in\mathbb{N}$. *Then* P *is a polynomial in the sums*

$$S_J = \sum_{k=1}^{\infty} \prod_{j\in J} x_{jk}, \qquad \emptyset \neq J\subset\{1,\ldots,d\}. \tag{7}$$

Proof. For $d=1$ we have $P=S_1$, and we shall proceed to general $d>1$ by induction. Thus assume that the corresponding quantity with products over $j\in\{1,\ldots,d-1\}$ is a polynomial P_{d-1} in the sums S_J with $\emptyset \neq J\subset\{1,\ldots,d-1\}$. Then we get, with $k_1,\ldots,k_d$ distinct throughout,

$$\begin{aligned}
\sum_{(k_j)} \prod_{j=1}^{d} x_{j,k_j} &= \sum_{k_1,\ldots,k_{d-1}} \prod_{j=1}^{d-1} x_{j,k_j} \sum_{k_d} x_{d,k_d} \\
&= \sum_{k_1,\ldots,k_{d-1}} \prod_{j=1}^{d-1} x_{j,k_j} (S_d - \sum_{i<d} x_{d,k_i}) \\
&= S_d \sum_{k_1,\ldots,k_{d-1}} \prod_{j=1}^{d-1} x_{j,k_j} - \sum_{i<d} \sum_{k_1,\ldots,k_{d-1}} x_{i,k_i} x_{d,k_i} \prod_{j\neq i,d} x_{j,k_j} \\
&= S_d P_{d-1}(S_J,\ J\subset\{1,\ldots,d-1\}) - \sum_{i<d} P_{d-1}(S_{J_i},\ J\subset\{1,\ldots,d-1\}),
\end{aligned}$$

where

$$J_i = \begin{cases} J, & i\notin J, \\ J\cup\{d\}, & i\in J. \end{cases}$$

Thus the statement remains true in the case of d factors. $\square$

Lemma 3.3. The assertions of Theorem 3.1 are true when η is non-random.

Proof. Writing π for an arbitrary partition of the set $\{1,\ldots,d\}$ into at most n subsets J, and (k_J) for an arbitrary collections of distinct indices k_J, $J\in\pi$, we get by the assumptions on ξ

$$E\prod_{j=1}^{d}\sum_{k=1}^{n}\xi_{jk}\eta_{jk} = \sum_{\pi}\sum_{(k_J)}E\prod_{J}\prod_{j\in J}\xi_{j,k_J}\eta_{j,k_J}$$

$$=\sum_{\pi}\sum_{(k_J)}\left\{E\prod_{J}\prod_{j\in J}\xi_{j,k_J}\right\}\left\{\prod_{J}\prod_{j\in J}\eta_{j,k_J}\right\}$$

$$=\sum_{\pi}\frac{(n-\#\pi)!}{n!}\left\{\sum_{(k_J)}\prod_{J}\prod_{j\in J}\xi_{j,k_J}\right\}\left\{\sum_{(k_J)}\prod_{J}\prod_{j\in J}\eta_{j,k_J}\right\}$$

which has the asserted form by Lemma 3.2.

For d=1, we get in particular

$$E_1 = E\sum_{k=1}^{n}\xi_k\eta_k = (E\xi_1)\sum_{k=1}^{n}\eta_k = n^{-1}R_1S_1.$$

To get E_2 and E_3 without effort when $R_j=S_j=0$, note that E_d is homogeneous of degree d in both the ξ_{jk} and the η_{jk}, so that necessarily

$$E_2 = c_2R_{12}S_{12},\qquad E_3 = c_3R_{123}S_{123},$$

for some constants c_2 and c_3. The latter may easily be obtained by direct computation in some simple example. We omit the details. □

Lemma 3.4. Let $\mathcal{F}$, ξ and η be such as in Theorem 3.1, except that the measure μ_ξ and the sums S_J are only assumed to be $\mathcal{F}_0$-measurable, the former with bounded support. Then

$$E[\prod_{j=1}^{d}\sum_{k=1}^{n}\xi_{jk}\eta_{jk}\,|\,\mathcal{F}_0] = P_{nd}\{R_J,S_J\}\quad \text{a.s.,} \tag{8}$$

for some polynomial P_{nd} in the sums R_J and S_J.

Proof. Let us first notice that the product in (8) is integrable by (C_1) and Hölder's inequality, so that the conditional expectations here and below exist. The statement of the lemma is trivially true for $n\wedge d=0$, if the product over an empty set is taken to be one. We shall proceed to general $n\geq 1$ by induction, so assume that the statement is true with n replaced by n-1 and for all d. Writing

J for an arbitrary subset of $\{1,\dots,d\}$, we get

$$E[\prod_{j=1}^{d}\sum_{k=1}^{n}\xi_{jk}\eta_{jk}|\mathcal{F}_0] = E[\sum_J \prod_{i\notin J}\xi_{i1}\eta_{i1}\prod_{j\in J}\sum_{k=2}^{n}\xi_{jk}\eta_{jk}|\mathcal{F}_0]$$

$$= \sum_J E[\prod_{i\notin J}\xi_{i1}\eta_{i1}\; E[\prod_{j\in J}\sum_{k=2}^{n}\xi_{jk}\eta_{jk}|\mathcal{F}_1]|\mathcal{F}_0]. \tag{9}$$

Now the sums

$$S_J' = \sum_{k=2}^{n}\prod_{j\in J}\eta_{jk} = S_J - \prod_{j\in J}\eta_{j1}, \qquad \emptyset \neq J\subset\{1,\dots,d\},$$

are $\mathcal{F}_0$-measurable, while the measure

$$\mu_\xi' = \sum_{k=2}^{n}\delta_{\xi_{.k}} = \mu_\xi - \delta_{\xi_{.1}}$$

is $\mathcal{F}_1$-measurable, so the induction hypothesis shows that the inner conditional expectations

$$E[\prod_{j\in J}\sum_{k=2}^{n}\xi_{jk}\eta_{jk}|\mathcal{F}_1], \qquad J\subset\{1,\dots,d\},$$

are polynomials in the sums S_I', $I\subset\{1,\dots,d\}$, as well as in the variables ξ_{jk} with $j=1,\dots,d$ and $k=2,\dots,n$. Thus the sum in (9) is a polynomial in the sums S_J, in the random variables $\eta_{11},\dots,\eta_{d1}$, and in the conditional product moments of the variables ξ_{jk}, given $\mathcal{F}_0$.

Let us now introduce an array $\xi'=(\xi_{jk}')$, by suitable randomization, such that ξ' is conditionally independent of $\mathcal{F}_n$, given $\mathcal{F}_0$, with the same conditional distribution as ξ. (This amounts to a randomization of the order between the vectors $\xi_{.1},\dots,\xi_{.n}$.) Write $\mathcal{F}_k'=\mathcal{F}_k\vee\sigma(\xi_{.1}',\dots,\xi_{.k}')$ for $k=0,\dots,n$, and note that the hypotheses of the lemma remain fulfilled for the triple $(\mathcal{F}',\xi',\eta)$. Repeating the above computation in the new situation, and noting that the result depends only on quantities which are the same in both cases, we get a.s.

$$E[\prod_{j=1}^{d}\sum_{k=1}^{n}\xi_{jk}\eta_{jk}|\mathcal{F}_0] = E[\prod_{j=1}^{d}\sum_{k=1}^{n}\xi_{jk}'\eta_{jk}|\mathcal{F}_0] = E[E[\prod_{j=1}^{d}\sum_{k=1}^{n}\xi_{jk}'\eta_{jk}|\mathcal{F}_n]|\mathcal{F}_0].$$

But Lemma 3.3 shows that the inner expectation on the right is a polynomial P_{nd} in the sums R_J and S_J. (Note that the sums R_J are the same for ξ and ξ'.) Since the latter are $\mathcal{F}_0$-measurable, this proves the assertion for sequences of length n, and hence completes the induction. □

This also completes the proof of Theorem 3.1, so we may turn to the case of infinite sequences. Here we shall write

$$\mu_J = E\prod_{j\in J}\xi_{j1}, \quad \mu'_J = E\prod_{j\in J}(\xi_{j1}-\mu_j), \quad S_J = \sum_{k=1}^{\infty}\prod_{j\in J}\eta_{jk}, \quad \emptyset \neq J\subset\{1,\ldots,d\},$$

whenever these quantities exist. Let us further introduce the condition

(C_2): There exist some constants $p_1,\ldots,p_d\geq 1$ with $\sum p_j^{-1}\leq 1$, such that

$$|E\xi_{j1}|\,E\sum_{k=1}^{\infty}|\eta_{jk}| + E|\xi_{j1}|^{p_j}\;E\Big\{\sum_{k=1}^{\infty}|\eta_{jk}|^{p'_j}\Big\}^{p''_j/2} < \infty, \qquad j=1,\ldots,d,$$

where $p'_j=p_j\wedge 2$ and $p''_j=p_j\vee 2$.

Theorem 3.5. Let $\mathcal{F}$ be a filtration on Z_+, and let ξ and η be infinite random sequences in R^d, such that ξ is $\mathcal{F}$-i.i.d. while η is $\mathcal{F}$-predictable. Let $\{K_1,\ldots,K_m\}$ be a partition of $\{1,\ldots,d\}$, such that the corresponding subarrays of ξ are independent. Assume that (C_2) is fulfilled, and that all products $\mu_j S_j$ are a.s. non-random, as well as all sums S_J with $2\leq\#J<d$ and $J\subset K_i$ for some i. Then

$$E_d = E\prod_{j=1}^{d}\sum_{k=1}^{\infty}\xi_{jk}\eta_{jk} = E\prod_{i=1}^{m}\sum_{\pi_i}\Big\{\prod_J \mu_J\Big\}P_{\pi_i}\{S_I\} = E\prod_{i=1}^{m}\sum_{\pi_i}\prod_J S_J\,P'_{\pi_i}\{\mu_I\}, \qquad (10)$$

where π_i denotes an arbitrary partition of K_i into subsets J, and where P_{π_i} and P'_{π_i} are polynomials in the sums S_I or moments μ_I, with I a union of sets in π_i or a subset of some $J\in\pi_i$, respectively. In particular $E_1=\mu_1 ES_1$, and under the further assumption $\mu_j S_j=0$ we have $E_2=\mu'_{12}ES_{12}$ and $E_3=\mu'_{123}ES_{123}$.

Weaker versions of this result may be obtained from Theorem 3.1 through a suitable approximation argument. However, a direct proof seems to be required to obtain the above statement in its full strength. A similar remark applies to the corresponding continuous time results in Theorems 4.1 and 5.1.

The theorem follows from Lemma 3.7 below. But first we need the result in a special case.

Lemma 3.6. The conclusions of Theorem 3.5 are true when η is non-random.

Proof. First conclude from (C_2), Proposition 2.1 and Hölder's inequality that E_d exists. From (C_2) it is further seen that S_j exists whenever $\mu_j\neq 0$.

Finally, it is seen from (C_2) and Hölder's inequality that S_J exists for all $J \subset \{1,\dots,d\}$ with $\#J \geq 2$, and that μ_J exists for all J. Similar arguments show that η can be approximated by a sequence with finitely many non-zero elements, so we may assume that already η has this form.

By independence, E_d splits into a product of similar expressions, with the products taken over the sets $K_1,\dots,K_m$. It is thus enough to consider each factor separately, so we may assume that $m=1$. Writing π for an arbitrary partition of $\{1,\dots,d\}$ into subsets J, and (k_J) for a corrsponding assignment of arbitrary distinct indices in N, we get

$$E\prod_{j=1}^{d}\sum_{k=1}^{\infty}\xi_{jk}\eta_{jk} = E\sum_{\pi}\sum_{(k_J)}\prod_{J}\prod_{j\in J}\xi_{j,k_J}\eta_{j,k_J} = \sum_{\pi}\Big\{\prod_J \mu_J\Big\}\Big\{\sum_{(k_J)}\prod_J\prod_{j\in J}\eta_{j,k_J}\Big\}.$$

By Lemma 3.2, the second factor on the right is a polynomial in the sums S_I with I a union of sets in π. This proves the first representation in (10), and the second one follows if instead we collect the terms corresponding to the different products $\prod S_I$. The explicit formula for E_1, and for E_2 and E_3 when $\mu_j=0$, are easily obtained in this case by direct computation. If instead $S_j=0$ for some j, we may reduce to the case $\mu_j=0$ by subtracting μ_j from each ξ_{jk}, which neither affects the sum $\sum\xi_{jk}\eta_{jk}$ nor the moments μ'_{12} and μ'_{123}. $\square$

<u>Lemma 3.7. Let $\mathcal{F}$, ξ and η be such as in Theorem 3.5, except that the products $\mu_j S_j$ as well as the sums S_J with $2 \leq \#J < d$ and $J \subset K_i$ for some i are only assumed to be $\mathcal{F}_0$-measurable. Then</u>

$$E[\prod_{j=1}^{d}\sum_{k=1}^{\infty}\xi_{jk}\eta_{jk}|\mathcal{F}_0] = E[\prod_{i=1}^{m}P_i|\mathcal{F}_0] \quad \text{a.s.,} \tag{11}$$

<u>where the P_i are polynomials of the stated form.</u>

<u>Proof.</u> First we note as before that the quantities involved in (11) exist because of (C_2). To prove (11), we shall proceed by induction over $d\in Z_+$, starting with the triviality $1=1$ for $d=0$. Thus we fix a $d\in N$, and assume that the statement is true for dimensions $<d$. Whenever S_J exists, write

$$U_{J,n} = \sum_{k>n} \prod_{j\in J} \eta_{jk} = S_J - \sum_{k\le n} \prod_{j\in J} \eta_{jk}, \quad n\in N,$$

and note that the sequence $(U_{J,n})$ is predictable. By the induction hypothesis, we may conclude that

$$E[\prod_{j\in J} \sum_{k>n} \xi_{jk}\eta_{jk} | \mathcal{F}_n] = \prod_{i=1}^{m} P_{J\cap K_i}(\mu_I, U_{I,n}) \quad \text{a.s.}, \quad n\in N,$$

where the factors on the right are polynomials in the products $\mu_j U_{j,n}$ with $j\in J\cap K_i$, in the moments μ_I with $I\subset J\cap K_i$, and in the sums $U_{I,n}$ with $I\subset J\cap K_i$ and $\# I\ge 2$. Letting J denote an arbitrary proper subset of $\{1,\ldots,d\}$, and conditioning in the n-th term below, first on $\mathcal{F}_n$ and then on $\mathcal{F}_{n-1}$, we obtain

$$\begin{aligned} E[\prod_{j=1}^{d} \sum_{k=1}^{\infty} \xi_{jk}\eta_{jk} | \mathcal{F}_0] &= E[\sum_J \sum_{n=1}^{\infty} \prod_{i\notin J} \xi_{in}\eta_{in} \prod_{j\in J} \sum_{k>n} \xi_{jk}\eta_{jk} | \mathcal{F}_0] \\ &= E[\sum_J \sum_{n=1}^{\infty} \prod_{i\notin J} \xi_{in}\eta_{in} \prod_{r=1}^{m} P_{J\cap K_r}(\mu_I, U_{I,n}) | \mathcal{F}_0] \\ &= E[\sum_J \mu_{J^c} \sum_{n=1}^{\infty} \prod_{i\notin J} \eta_{in} \prod_{r=1}^{m} P_{J\cap K_r}(\mu_I, U_{I,n}) | \mathcal{F}_0]. \end{aligned}$$

The remaining argument is similar to that in Lemma 3.4. Thus we construct some $\xi' \overset{d}{=} \xi$ independent of $\mathcal{F}_\infty = \bigvee \mathcal{F}_n$, and put $\mathcal{F}'_k = \mathcal{F}_k \vee \sigma(\xi'_{\cdot 1},\ldots,\xi'_{\cdot k})$ for $k\in Z_+$. Since the above computation gives the same result for the triple $(\mathcal{F}', \xi', \eta)$, we get

$$E[\prod_{j=1}^{d} \sum_{k=1}^{\infty} \xi_{jk}\eta_{jk} | \mathcal{F}_0] = E[\prod_{j=1}^{d} \sum_{k=1}^{\infty} \xi'_{jk}\eta_{jk} | \mathcal{F}_0] = E[E[\prod_{j=1}^{d} \sum_{k=1}^{\infty} \xi'_{jk}\eta_{jk} | \mathcal{F}_\infty] | \mathcal{F}_0] \quad \text{a.s.}$$

Here the right-hand side has the desired form by Lemma 3.6, which completes the induction. $\square$

4. Processes on [0,1]

In this section we shall prove our decoupling identity for stochastic integrals with respect to exchangeable processes on [0,1]. So we consider R^d-valued processes $X=(X_1,\ldots,X_d)$ and $V=(V_1,\ldots,V_d)$ on [0,1], where X is ergodic exchangeable and directed by (α,ρ,β). Put $\sigma_j=\rho_{jj}^{1/2}$, and define for non-empty $J\subset\{1,\ldots,d\}$

$$\beta_{Jk}=\prod_{j\in J}\beta_{jk},\quad \beta_J=\sum_{k=1}^{\infty}\delta_{\beta_{Jk}},\quad B_J=\sum_{k=1}^{\infty}\beta_{Jk}, \tag{1}$$

$$V_J=\prod_{j\in J}V_j,\quad S_J=\int_0^1 V_J,\quad V_J'=\prod_{j\in J}(V_j-S_j),\quad S_J'=\int_0^1 V_J', \tag{2}$$

whenever these expressions make sense. Note that β_J is to be regarded as a measure on $R\setminus\{0\}$. We shall need the following condition.

(C_3): There exist some constants $p_j\geq 1$ and $q_j>2p_j$, $j=1,\ldots,d$, with $\sum p_j^{-1}\leq 1$ and such that for every j

$$|\alpha_j|E\int_0^1|V_j|+\sigma_jE\left\{\int_0^1 V_j^2\right\}^{p_j/2}+\sum_{k=1}^{\infty}|\beta_{jk}|^{p_j}\ E\int_0^1|V_j|^{q_j}<\infty.$$

Theorem 4.1. Let $\mathcal{F}$ be a standard filtration on [0,1], and let X and V be R^d-valued processes on [0,1], such that X is ergodic $\mathcal{F}$-exchangeable and directed by (α,ρ,β) while V is $\mathcal{F}$-predictable. Assume that (C_3) is fulfilled, and that all the products α_jS_j, $\rho_{ij}S_{ij}$ and β_JS_J are a.s. non-random. Then

$$E_d=E\prod_{j=1}^{d}\int_0^1 V_j\,dX_j=\sum_{\pi}\left\{\prod_i\alpha_iS_i\right\}\left\{\prod_{j,k}\rho_{jk}S_{jk}'\right\}\left\{\prod_J B_J\right\}P_\pi\{S_I'\}, \tag{3}$$

where the summation extends over all partitions π of $\{1,\ldots,d\}$ into singletons $\{i\}$, pairs $\{j,k\}$, and sets J with $\#J\geq 2$, and where P_π is a polynomial in the integrals S_I' with I a subset of some $J\in\pi$. In particular $E_1=\alpha_1S_1$, and under the further assumption that $\alpha_jS_j=0$ we have $E_2=(\rho_{12}+B_{12})S_{12}'$ and $E_3=B_{123}S_{123}'$.

As in case of Theorem 3.5, there is also a dual form of (3), with B and S interchanged in the last two factors, and with each P_π a polynomial in the sums B_I with I a union of sets $J\in\pi$. When X has finite variation, it is natural to write $\alpha_j'=\alpha_j-B_j$, and to replace (3) by

$$E_d = E\prod_{j=1}^{d}\int_0^1 V_j dX_j = \sum_{\pi}\left\{\prod_j \alpha_j' S_j\right\}\left\{\prod_J B_J\right\} P_\pi\{S_I\}, \tag{4}$$

where π denotes an arbitrary partition of $\{1,\ldots,d\}$ into singletons $\{i\}$ and sets J, and where P_π is a polynomial in the integrals S_I with I a subset of some $J\in\pi$.

For the proof of Theorem 4.1 we shall need two lemmas, where the first one will also be needed later. Let us write $M^*=\sup_t|M_t|$.

Lemma 4.2. Let $M_1,\ldots,M_d$ be continuous $\mathcal{F}$-martingales starting at 0, and such that $\rho_{ij}=[M_i,M_j]_\infty$ is a.s. non-random for $i\neq j$. Further assume that $\|M_j^*\|_{p_j}<\infty$ for some constants $p_j\geq 1$, $j=1,\ldots,d$, where $p^{-1}=\sum p_j^{-1}\leq 1$. Then the martingale

$$M_t = E[\prod_{j=1}^{d} M_j(\infty)|\mathcal{F}_t], \qquad t\geq 0, \tag{5}$$

has a continuous version satisfying $\|M^*\|_p<\infty$, and moreover

$$E\, M_t = M_0 = \sum_\pi \prod_{i,j}\rho_{ij} \quad \text{a.s.}, \tag{6}$$

where the summation extends over all partitions π of the set $\{1,\ldots,d\}$ into pairs $\{i,j\}$.

Proof. Write

$$V_J(t) = [M_i,M_j]_t, \qquad \rho_J = V_J(\infty), \qquad J=\{i,j\}\subset\{1,\ldots,d\},$$

and conclude from Itô's formula that

$$\prod_{j=1}^{d} M_j(t) = \sum_{j=1}^{d}\int_0^t \prod_{i\neq j} M_i dM_j + \sum_J \int_0^t \prod_{i\notin J} M_i dV_J, \qquad t\geq 0, \tag{7}$$

where the last summation extends over all (unordered) pairs $J\subset\{1,\ldots,d\}$. Applying this formula to the integrands in the last sum and proceeding recursively, we get

$$\begin{aligned}\prod_j M_j = \sum_j \int \prod_i M_i dM_j &+ \sum_{1\leq k<d/2}\sum_{J_1}\cdots\sum_{J_k}\sum_j \int dV_{J_1}\int\cdots\int dV_{J_k}\int\prod_i M_i dM_j \\ &+ \sum_{J_1}\cdots\sum_{J_{d/2}}\int dV_{J_1}\int\cdots\int dV_{J_{d/2}},\end{aligned} \tag{8}$$

where the last sum occurs only when d is even. Here the summations in the k-th term extend over all sequences of disjoint pairs $J_1,\ldots,J_k\subset\{1,\ldots,d\}$ and over remaining indices j, while the product in the integrand extends over all indices $i\neq j$ outside $J_1,\ldots,J_k$. Finally, the integration is taken over the set

$\{(t_1,\dots,t_{k+1})\in R_+^{k+1};\ t_1\geq t_2\geq\dots\geq t_{k+1}\geq 0\}$. Similar conventions apply to the last sum in (8).

We shall now use the fact that, if $V_1,\dots,V_k$ are continuous functions of bounded variation starting at 0, then

$$\sum_r \int dV_{r_1}\int\dots\int dV_{r_k} = \prod_{j=1}^k V_j,$$

where the summation extends over all permutations $r=(r_1,\dots,r_k)$ of $(1,\dots,k)$. Applying this to (8), we get

$$\prod_j M_j = \sum_j \int \prod_i M_i dM_j + \sum_{1\leq k<d/2} \sum_{\pi'} \sum_j \int d\prod_J V_J \int \prod_i M_i dM_j + \sum_{\pi} \prod_J V_J, \tag{9}$$

where the inner summations in the k-th term extend over all (unordered) collections π' of k disjoint pairs $J\subset\{1,\dots,d\}$, and over remaining indices j, while the last product is taken over all other indices i. Moreover, integration is now over the set $\{(t_1,t_2)\in R_+^2;\ t_1\geq t_2\}$.

Next we integrate by parts in (9) to obtain

$$\prod_j M_j = \sum_j \int \prod_i M_i dM_j + \sum_{1\leq k<d/2} \sum_{\pi'} \sum_j \Big\{\prod_J V_J \int \prod_i M_i dM_j - \int \prod_J V_J \prod_i M_i dM_j\Big\} + \sum_{\pi} \prod_J V_J.$$

Changing the order of summation and noting that the products $\prod V_J(\infty)=\prod \rho_J$ are a.s. constant, we finally obtain

$$\prod_j M_j(\infty) = \sum_j \sum_{\pi'} \int_0^\infty \Big\{\prod_J \rho_J - \prod_J V_J\Big\} \prod_i M_i dM_j + \sum_{\pi} \prod_J \rho_J, \tag{10}$$

where the inner summation in the first term extends over all partitions π' of the set $\{1,\dots,d\}\setminus\{j\}$ into pairs J plus a remaining set of indices i. The assertions of the lemma will follow immediately from (10), if we can only prove that the integral processes on the right are bounded by random variables in L_p.

To see this, let $p_J^{-1}=p_i^{-1}+p_j^{-1}$ when $J=\{i,j\}$, and note by the BDG and Hölder inequalities that

$$\|V_J^*\|_{p_J} \leq \|\{[M_i,M_i][M_j,M_j]\}^{1/2}\|_{p_J} \leq \|[M_i,M_i]^{1/2}\|_{p_i} \|[M_j,M_j]^{1/2}\|_{p_j} \lesssim \|M_i^*\|_{p_i} \|M_j^*\|_{p_j}.$$

By the same inequalities, we hence obtain for fixed j and π' as above

$$\Big\|\sup_t \Big|\int_0^t \Big\{\prod_J \rho_J - \prod_J V_J\Big\} \prod_i M_i dM_j\Big|\Big\|_p \lesssim \Big\|\Big\{\int_0^\infty \Big\{\prod_J \rho_J - \prod_J V_J\Big\}^2 \prod_i M_i^2 d[M_j,M_j]\Big\}^{1/2}\Big\|_p$$

$$\le \left\| \prod_j V_j^* \prod_i M_i^* \, [M_j,M_j]_\infty^{1/2} \right\|_p \le \prod_j \|V_j^*\|_{p_J} \prod_i \|M_i^*\|_{p_i} \|[M_j,M_j]^{1/2}\|_{p_j} \lesssim \prod_{k=1}^d \|M_k^*\|_{p_k} < \infty . \square$$

Lemma 4.3. *Let* M *be a continuous* $\mathcal{F}$-*martingale on* $[0,1]$, *let* $V_1,\ldots,V_d$ $(d\ge 1)$ *be* $\mathcal{F}$-*predictable processes on* $[0,1]$ *with* $\int V_1 = \ldots = \int V_d = 0$ *a.s., and let* $\tau_1,\ldots,\tau_d$ *be i.i.d.* $U(0,1)$ *random variables such that the processes* $1\{\tau_j \le t\}$ *are* $\mathcal{F}$-*exchangeable. Assume for some constants* $p,q_1,\ldots,q_d \ge 1$ *with* $p^{-1}+2\sum q_j^{-1} \le 1$ *that*

$$E|M^*|^p < \infty; \qquad E\int |V_j|^{q_j} < \infty, \qquad j=1,\ldots,d. \tag{11}$$

Then

$$E\, M_1 \prod_{j=1}^d V_j(\tau_j) = 0. \tag{12}$$

Proof. By Proposition 2.6 we have

$$V_j(\tau_j) = \int_0^1 V_j(t)\, d(1\{\tau_j \le t\}) = \int_0^1 U_j\, dM_j, \qquad j=1,\ldots,d, \tag{13}$$

where the martingales M_j and the predictable processes U_j are given by

$$M_j(t) = 1\{\tau_j \le t\} - \log(1 - t \wedge \tau_j), \qquad t\in[0,1], \tag{14}$$

$$U_j(t) = V_j(t) - \frac{1}{1-t}\int_t^1 V_j(s)\,ds, \qquad t\in[0,1). \tag{15}$$

Since the martingale M and the integral processes $N_j = \int U_j\, dM_j$ are mutually orthogonal, we get by Itô's formula

$$M(t) \prod_{j=1}^d N_j(t) = \int_0^t \prod_j N_j\, dM + \sum_j \int_0^{t+} M \prod_{i\ne j} N_i\, dN_j \quad \text{a.s.}, \quad t\in[0,1]. \tag{16}$$

Thus (12) will follow if we can show that the integral processes on the right are martingales.

To see this, choose $p_j < q_j/2$, $j=1,\ldots,d$, such that $p^{-1} + \sum p_j^{-1} = 1$. Using the BDG and Hölder inequalities plus Proposition 2.8, we get from (11)

$$E \sup_t \left| \int_0^t \prod_j N_j\, dM \right| \lesssim E\left\{ \int_0^1 \prod_j N_j^2\, d[M,M] \right\}^{1/2} \le E\, [M,M]_1^{1/2} \prod_j N_j^*$$

$$\le \|[M,M]_1^{1/2}\|_p \prod_j \|N_j^*\|_{p_j} \lesssim \|M^*\|_p \prod_j \left\{ E\int_0^1 |V_j|^{q_j} \right\}^{1/q_j} < \infty,$$

so by uniform integrability the integral on the left must be a martingale. In the same way we get for $j=1,\ldots,d$

$$E \sup_t \left| \int_0^t M \prod_{i\neq j} N_i dN_j \right| \lesssim E\left\{ \int_0^1 M^2 \prod_{i\neq j} N_i^2 d[N_j, N_j] \right\}^{1/2} < \infty,$$

where the finiteness of the second expression shows that the stochastic integral on the left is a local martingale, and hence justifies the use of the BDG inequality in the first step. □

Proof of Theorem 4.1. Note first that E_d exists, in view of (C_2) and Propositions 2.6 and 2.8. By Proposition 2.4 we may integrate termwise in (1.5), and by Proposition 2.8 it is enough to assume that X has finitely many jumps. Writing $X_j'(t) \equiv X_j(t) - \alpha_j t$, so that $X_j'(1) \equiv 0$, we get

$$E \prod_{j=1}^{d} \int_0^1 V_j dX_j = E \prod_{j=1}^{d} \left\{ \alpha_j S_j + \int_0^1 V_j' dX_j' \right\} = \sum_J \prod_{j\notin J} \alpha_j S_j \; E \prod_{j\in J} \int_0^1 V_j' dX_j',$$

where the summation extends over all subsets $J \subset \{1,\ldots,d\}$. Thus we may further assume that $\alpha_j \equiv S_j \equiv 0$.

For each $j=1,\ldots,d$ we write M_j for the martingale component of B_j and define U_j by (15), so that

$$\int_0^1 V_j dB_j = \int_0^1 U_j dM_j = N_j(1), \quad j=1,\ldots,d, \tag{17}$$

by Proposition 2.6, where the integral process N_j on the right is a continuous L_{q_j}-martingale. From Lemma 2.7 it is further seen that for $i\neq j$

$$[N_i, N_j]_1 = \int_0^1 U_i U_j d[M_i, M_j] = \rho_{ij} \int_0^1 U_i U_j = \rho_{ij} \int_0^1 V_i V_j = \rho_{ij} S_{ij}.$$

By Lemma 4.2 there hence exists for every $J \subset \{1,\ldots,d\}$ some continuous L_{p_J}-martingale M_J $(p_J^{-1} = \sum_J p_j^{-1})$ satisfying

$$M_J(1) = \prod_{j\in J} N_j(1) = \prod_{j\in J} \int_0^1 V_j dB_j, \tag{18}$$

and

$$E\, M_J(1) = \sum_{\pi_J} \prod_{i,j} [N_i, N_j]_1 = \sum_{\pi_J} \prod_{i,j} \rho_{ij} S_{ij}, \tag{19}$$

where the summations in (19) extend over all partitions π_J of J into pairs $\{i,j\}$.

Let us now write π' for an arbitrary collection of disjoint sets $J \subset \{1,\ldots,d\}$, put $J' = \bigcap J^c$, and let the indices $k_J \in N$, $J \in \pi'$, be different but otherwise arbitrary. Using Proposition 2.4 and (18), we get

$$E\prod_{j=1}^{d}\int_0^1 V_j\,dX_j = E\prod_{j=1}^{d}\left\{\int_0^1 V_j\,dB_j + \sum_{k=1}^{\infty}\beta_{jk}V_j(\tau_k)\right\}$$

$$= E\sum_{\pi'} M_{J'}(1)\sum_{(k_J)}\prod_J \beta_{J,k_J}V_J(\tau_{k_J}) = \sum_{\pi'}\sum_{(k_J)}\prod_J \beta_{J,k_J} E\, M_{J'}(1)\prod_J V_J(\tau_{k_J}).$$

Writing

$$\prod_J V_J(\tau_{k_J}) = \prod_J\left\{S_J + (V_J(\tau_{k_J})-S_J)\right\},$$

and expanding the product on the right, it is seen from Lemma 4.3 and (19) that

$$E\, M_{J'}(1)\prod_J V_J(\tau_{k_J}) = E\, M_{J'}(1)\prod_J S_J = \sum_{\pi_{J'}}\prod_{i,j}\rho_{ij}S_{ij}\prod_J S_J,$$

so we get

$$E\prod_{j=1}^{d}\int_0^1 V_j\,dX_j = \sum_{\pi}\prod_{i,j}\rho_{ij}S_{ij}\prod_J S_J\sum_{(k_J)}\prod_J \beta_{J,k_J}. \tag{20}$$

By Lemma 3.2, the inner sum on the right is a polynomial in the sums B_K with K a union of sets $J\in\pi'$. If instead we collect the terms involving a given product $\prod B_K$, it is clear that the coefficient will be a polynomial in the integrals S_J with J a subset of some K. This completes the proof of (3).

The explicit formula for E_1 follows immediately from (3), while those for E_2 and E_3 *when* $\alpha_j S_j = 0$ *are obtained from* (20) *with* S'_J *in place of* S_J. □

5. Processes on R_+

Our aim in this section is to prove a decoupling identity for stochastic integrals with respect to Lévy processes. Thus we consider R^d-valued processes $X=(X_1,\ldots,X_d)$ and $V=(V_1,\ldots,V_d)$ on R_+, where X is Lévy and directed by (γ,ρ,ν). Put $\sigma_j=\rho_{jj}^{1/2}$, and define for non-empty subsets $J\subset\{1,\ldots,d\}$

$$N_J = \int\prod_{j\in J}x_j\,\nu(dx) = \int\prod_{j\in J}x_j\,\nu(dx_1\ldots dx_d),\qquad V_J = \prod_{j\in J}V_j,\qquad S_J = \int_0^\infty V_J, \tag{1}$$

whenever these expressions make sense. The following condition will be needed.

(C_4): There exist some constants $p_1,\ldots,p_d\geq 1$ with $\sum p_j^{-1}\leq 1$, such that for all j

$$|\gamma_j|\,E\int|V_j| + \sigma_j E\left\{\int V_j^2\right\}^{p_j/2} + \int|x_j|^{p_j}\nu(dx)\;E\left[\left\{\int|V_j|^{p_j'}\right\}^{p_j''/2}+\int|V_j|^{p_j}\right] < \infty,$$

where $p_j'=p_j\wedge 2$ and $p_j''=p_j\vee 2$.

<u>Theorem 5.1. Let $\mathcal{F}$ be a standard filtration on R_+, and let X and V be R^d-valued processes on R_+, such that X is $\mathcal{F}$-Lévy and directed by (γ,ρ,ν) while V is $\mathcal{F}$-predictable. Assume that (C_4) is fulfilled, and that the products $\gamma_j S_j$ (for $d>1$), $\rho_{ij}S_{ij}$ (for $d>2$), and $N_J S_J$ (for $2\leq \# J<d$) are a.s. non-random. Then</u>

$$E\prod_{j=1}^{d}\int_0^\infty V_j\,dX_j = E\sum_{\pi}\prod_i \gamma_i S_i \prod_{j,k}(\rho_{jk}+N_{jk})S_{jk}\prod_J N_J S_J, \tag{2}$$

<u>where the summation extends over all partitions π of $\{1,\ldots,d\}$ into singletons $\{i\}$, pairs $\{j,k\}$, and subsets J with $\# J\geq 3$.</u>

Note that N_{jk} can be omitted from the second product on the right, provided that sets J with $\# J=2$ are allowed in π. If X has locally finite variation while $|V|$ is integrable on R_+, one may introduce the constants $\gamma_j'=\gamma_j-N_j$, and write (2) in the form

$$E\prod_{j=1}^{d}\int_0^\infty V_j\,dX_j = E\sum_{\pi}\prod_i \gamma_i' S_i\prod_J N_J S_J, \tag{3}$$

where the summation extends over all partitions π of $\{1,\ldots,d\}$ into singletons $\{i\}$ and subsets J.

The method of proof is similar to that for Theorem 3.5, though technically more complicated. The key step is Lemma 5.8, where we proceed by induction over d to establish a conditional version of (2) (though formally in terms of optional

projections). Our proof of Lemma 5.8 requires ν to be bounded, so a reduction to that case is given through the construction in Lemmas 5.4 and 5.5. We shall also need some simple moment estimates, as provided by Lemmas 5.2 and 5.3. The remaining Lemmas 5.6, 5.7 and 5.9 are simple results in real analysis and stochastic calculus, which ought to be known, though we were unable to find references.

Unless otherwise stated, we assume that X and V are such as in Theorem 5.1, and in particular that (C_4) is fulfilled. As before, let p_J be defined for subsets $J \subset \{1,\dots,d\}$ by $p_J^{-1} = \sum_J p_j^{-1}$.

Lemma 5.2. For any $J \subset \{1,\dots,d\}$ with $\#J \geq 2$, we have

$$\int \prod_{j\in J} |x_j|^p \nu(dx) \quad E\left\{\int |V_J|^p\right\}^{p_J/p} < \infty, \qquad 1 \leq p \leq p_J. \tag{4}$$

Proof. We may assume that

$$\int |x_j|^{p_j} \nu(dx) < \infty, \qquad E\left\{\int |V_j|^{p_j'}\right\}^{p_j''/2} + E\int |V_j|^{p_j} < \infty, \qquad j \in J, \tag{5}$$

since (4) is trivially true if any of these integrals vanishes. Then Hőlder's inequality yields

$$\int \prod_{j\in J} |x_j|^{p_J} \nu(dx) < \infty, \qquad E\int |V_J|^{p_J} < \infty,$$

so by norm interpolation (formula (2.10)) it remains to show that

$$\int \prod_{j\in J} |x_j| \, \nu(dx) < \infty, \qquad E\left\{\int |V_J|\right\}^{p_J} < \infty.$$

To see this, note that $x^{p\wedge 2} \leq x^2 \wedge 1 + x^p$ for $x,p > 0$, so that by (5)

$$\int |x_j|^{p_j'} \nu(dx) \leq \int (x_j^2 \wedge 1) \nu(dx) + \int |x_j|^{p_j} \nu(dx) < \infty, \qquad j \in J. \tag{6}$$

By norm interpolation we get from (5) and (6)

$$\int |x_j|^p \nu(dx) < \infty, \qquad E\left\{\int |V_j|^p\right\}^{p_j/p} < \infty, \qquad p_j' \leq p \leq p_j, \quad j \in J. \tag{7}$$

Now clearly

$$\sum_{j\in J} p_j^{-1} = p_J^{-1} \leq 1 \leq \tfrac{1}{2}\#J \leq \sum_{j\in J} p_j'^{-1},$$

so we may choose some $q_j \in [p_j', p_j]$, $j \in J$, satisfying $\sum_J q_j^{-1} = 1$. Using (7) with $p = q_j$, we get by Hőlder's inequality

$$\int \prod_{j\in J} |x_j| \nu(dx) \le \prod_{j\in J} \left\{ \int |x_j|^{q_j} \nu(dx) \right\}^{1/q_j} < \infty,$$

$$E\left\{ \int |V_J| \right\}^{p_J} \le E \prod_{j\in J} \left\{ \int |V_j|^{q_j} \right\}^{p_J/q_j} \le \prod_{j\in J} \left\{ E\left\{ \int |V_j|^{q_j} \right\}^{p_j/q_j} \right\}^{p_J/p_j} < \infty,$$

as desired. □

In the special case when $\rho=0$, we introduce the covariation processes X_J and their associated total variation processes $\bar{X}_J$, given for $J \subset \{1,\ldots,d\}$ with $\# J \ge 2$ and for $t \ge 0$ by

$$X_J(t) = \sum_{s\le t} \prod_{j\in J} \Delta X_j(s), \quad \bar{X}_J(t) = \sum_{s\le t} \prod_{j\in J} |\Delta X_j(s)| = \int_0^t |dX_J|. \tag{8}$$

Note that X_J and $\bar{X}_J$ are again $\mathcal{F}$-Lévy, with Lévy measures ν_J and $\bar{\nu}_J$, given for Borel sets $B \subset R \setminus \{0\}$ by

$$\nu_J(B) = \nu\left\{x\in R^d;\ \prod_{j\in J} x_j \in B\right\}, \quad \bar{\nu}_J(B) = \nu\left\{x\in R^d;\ \prod_{j\in J} |x_j| \in B\right\}. \tag{9}$$

In particular X_J has drift N_J. Recall that $p_J^{-1} = \sum_J p_j^{-1}$.

<u>Lemma 5.3.</u> <u>If</u> $\rho=0$, <u>we have for any</u> $J \subset \{1,\ldots,d\}$ <u>with</u> $\# J \ge 2$

$$E\left\{ \int |V_J dX_J| \right\}^{p_J} \lesssim \sup_{1\le p\le p_J} \left\{ \int \prod_{j\in J} |x_j|^p \nu(dx) \right\}^{p_J/p} \sup_{1\le p\le p_J} E\left\{ \int |V_J|^p \right\}^{p_J/p} < \infty. \tag{10}$$

<u>Proof.</u> The expression on the right is finite by Lemma 5.2 plus norm interpolation, so Proposition 2.2 applies with X, V and p replaced by $\bar{X}_J$, $|V_J|$ and p_J, and (10) follows. □

<u>Lemma 5.4.</u> <u>Fix arbitrary numbers</u> $m_J \in R$, $\emptyset \ne J \subset \{1,\ldots,d\}$. <u>Then there exists some measure</u> μ <u>on the cube</u> $C=[-1,1]^d$, <u>such that</u> $\mu(C) = \sum |m_J|$ <u>and moreover</u>

$$\int \prod_{j\in J} x_j \mu(dx) = m_J, \quad \emptyset \ne J \subset \{1,\ldots,d\}. \tag{11}$$

<u>Proof.</u> Suppose we can find some probability measures μ_J^+ and μ_J^- on C satisfying

$$\int \prod_{i\in I} x_i \mu_J^{\pm}(dx) = \pm 1\{I=J\}, \quad \emptyset \ne I, J \subset \{1,\ldots,d\}. \tag{12}$$

Then the measure

$$\mu = \sum_J (m_J \vee 0)\mu_J^+ - \sum_J (m_J \wedge 0)\mu_J^- \tag{13}$$

has clearly the desired properties. To construct $\mu_J^{\pm}$, fix $k\in J$, and let ξ_j, $j\in J\setminus\{k\}$,

be independent random variables (on some probability space) with $P\{\xi_j=1\}=P\{\xi_j=-1\}=1/2$. Choose ξ_k such that $\prod_J \xi_j = \pm 1$, and let $\xi_j=0$ for $j \notin J$. Take $\mu_J^{\pm}$ to be the distribution of $(\xi_1,\ldots,\xi_d)$. Then (12) is trivially fulfilled for $I \setminus J \neq \emptyset$ or $I \subset J\setminus\{k\}$, and if $k \in I \subset J$ we get

$$\int \prod_{i\in I} x_i \mu_J^{+}(dx) = E\prod_{i\in I}\xi_i = \pm E \prod_{i\in I}\xi_i \prod_{j\in J}\xi_j = \pm E \prod_{j\in J\setminus I}\xi_j = \pm 1\{I=J\}. \qquad \square$$

For the purpose of the next lemma, say that the probability space $(\Omega', \mathcal{O}', P')$ is an <u>extension</u> of $(\Omega, \mathcal{O}, P)$, if P is the image of P' under some $\mathcal{O}'/\mathcal{O}$-measurable mapping $\psi: \Omega' \to \Omega$. Note that any random element ξ on Ω then extends, with preserved distributional and path properties, to a random element ξ' on Ω' through the composition $\xi' = \xi \circ \psi$. We shall further say that a filtration $\mathcal{F}'$ on Ω' extends $\mathcal{F}$ on Ω, if ψ is also $\mathcal{F}'_t/\mathcal{F}_t$-measurable for every t. In this case, adaptedness and predictability are automatically preserved by the extension, as is the stopping time property of a random variable. Usually $(\Omega', \mathcal{O}', P')$ is formed as a product of $(\Omega, \mathcal{O}, P)$ with some other probability space, in which case ψ is always taken to be the natural projection of Ω' onto Ω. (Cf. [4], p. 89.)

<u>Lemma 5.5. For every</u> $\varepsilon > 0$ <u>there exists, on some extended standard filtered probability space</u> $(\Omega', \mathcal{O}', \mathcal{F}', P')$, <u>an</u> R^d-<u>valued</u> $\mathcal{F}'$-<u>Lévy process</u> X' <u>on</u> R_+, <u>such that</u> X' <u>is directed by</u> (γ, ρ, ν') <u>for some bounded and boundedly supported Lévy measure</u> ν' <u>with the same moments</u> N_J $(\#J \geq 2)$ <u>as</u> ν, <u>and such that moreover</u>

$$E\left|\prod_{j=1}^{d}\int_0^{\infty} V_j dX'_j - \prod_{j=1}^{d}\int_0^{\infty} V_j dX_j\right| < \varepsilon. \tag{14}$$

<u>Proof.</u> For each $n \in N$, form a process Y_n on R_+ by adding to the drift and diffusion components of X the centered sum of jumps in X with size between n^{-1} and n. Note that both Y_n and $X-Y_n$ are again $\mathcal{F}$-Lévy, and directed by $(\gamma, \rho, \varkappa_n)$ and $(0,0,\varkappa'_n)$, respectively, where $\varkappa_n$ is the restriction of ν to the set $\{x \in R^d;\ n^{-1} < |x| < n\}$, while $\varkappa'_n = \nu - \varkappa_n$. For $J \subset \{1,\ldots,d\}$ with $\#J \geq 2$, put

$$m_{nJ} = \int \prod_{j\in J} x_j \varkappa'_n(dx) = N_J - \int \prod_{j\in J} x_j \varkappa_n(dx), \tag{15}$$

and form a measure μ_n as in Lemma 5.4. Next define on the Lebesgue unit interval

$(I,\mathcal{B},\lambda)$ a centered compound Poisson process Z_n with Lévy measure μ_n, and consider all random processes as functions on the product space $(\Omega', \mathcal{O}', P')$ $=(\Omega \times I, \mathcal{O}\times\mathcal{B}, P\times\lambda)$. Let $\mathcal{F}_n$ be the $(\mathcal{O}', P')$-completed filtration on Ω' generated by $\mathcal{F}$ and Z_n, and note that $\mathcal{F}_n$ is automatically right-continuous. It is easy to check that the pair (Y_n, Z_n) and hence also the sum $U_n=Y_n+Z_n$ are $\mathcal{F}_n'$-Lévy. Note also that U_n is directed by (γ,ρ,ν_n), where $\nu_n = \varkappa_n+\mu_n$, and that ν_n gives the same values as ν to the moments N_J with $\#J\geq 2$.

It remains to show that (14) is fulfilled for $X'=U_n$ when n is large. We may then assume by (C_4) and Lemma 5.2 that (with $p_j'=p_j\wedge 2$)

$$\int |x_j|^p \nu(dx) < \infty, \quad p\in[p_j',p_j], \quad j=1,\ldots,d, \tag{16}$$

$$\int \prod_{j\in J} |x_j| \,\nu(dx) < \infty, \quad J\subset\{1,\ldots,d\}, \quad \#J\geq 2. \tag{17}$$

From (15)-(17) we get by dominated convergence as $n\to\infty$

$$\int |x_j|^p \varkappa_n'(dx) \to 0, \qquad p\in[p_j',p_j], \qquad j=1,\ldots,d, \tag{18}$$

$$\int |x_j|^p \mu_n(dx) \leq \mu_n(R^d) = \sum_J |m_{nJ}| \leq \int \prod_{j\in J}|x_j|\,\varkappa_n'(dx) \to 0. \tag{19}$$

Hence, by (C_4) and Proposition 2.2,

$$\left\| \int_0^\infty V_j\, d(X_j-U_{nj}) \right\|_{p_j} \leq \left\| \int_0^\infty V_j\, d(X_j-Y_{nj}) \right\|_{p_j} + \left\| \int_0^\infty V_j\, dZ_{nj} \right\|_{p_j} \to 0. \tag{20}$$

Since also $\left\| \int V_j\, dX_j \right\|_{p_j} < \infty$ for each j, we may conclude by Hölder's inequality that the left-hand side of (14) tends to zero as $n\to\infty$. □

Lemma 5.6. Let $F_1,\ldots,F_d$ be right-continuous functions of locally bounded variation, and define for $J\subset\{1,\ldots,d\}$ with $\#J\geq 2$

$$F_J(t)-F_J(s) = \sum_{u\in(s,t]} \prod_{j\in J} \Delta F_j(u), \qquad -\infty < s \leq t < \infty. \tag{21}$$

Then

$$\prod_{j=1}^d F_j(t) - \prod_{j=1}^d F_j(s) = \sum_J \int_{s+}^{t+} dF_J(u) \prod_{j\notin J} F_j(u-), \qquad -\infty < s \leq t < \infty. \tag{22}$$

where the summation extends over all non-empty subsets $J\subset\{1,\ldots,d\}$.

Proof. This is obvious for d=1, and for d=2 it reduces to the formula for integration by parts. The assertion for general d follows easily by induction. □

Lemma 5.7. Let X, Y and A be random processes on R_+, such that X is measurable with $EX^* < \infty$, while Y is optional and A is adapted and right-continuous with locally bounded variation. Assume that $E[X_\tau; \tau<\infty]=E[Y_\tau; \tau<\infty]$ for every stopping time τ, and that $E\int|X|\,|dA|<\infty$. Then $E\int XdA=E\int YdA$.

Proof. For any stopping time τ,

$$Y_\tau = E[X_\tau | \mathcal{F}_\tau] \quad \text{a.s. on } \{\tau<\infty\},$$

so by Jensen's inequality

$$E[|Y_\tau|; \tau<\infty] \le E[|X_\tau|; \tau<\infty].$$

Assuming without loss that A is non-decreasing, and letting τ_t, $t\ge 0$, denote the associated random time change, we get

$$E\int|Y|dA = \int E[|Y_{\tau_t}|; \tau_t<\infty]dt \le \int E[|X_{\tau_t}|; \tau_t<\infty]dt = E\int|X|dA.$$

Using Fubini's theorem, we thus obtain

$$E\int YdA = \int E[Y_{\tau_t}; \tau_t<\infty]dt = \int E[X_{\tau_t}; \tau_t<\infty]dt = E\int XdA. \qquad \square$$

We are now ready for our key lemma, where we assume again that X and V are such as in Theorem 5.1. For any subset $J\subset\{1,\dots,d\}$ with $\#J\ge 2$, we write

$$U_J(t) = \int_t^\infty V_J(s)ds, \qquad t\ge 0. \tag{23}$$

Lemma 5.8. Assume that $\gamma,\rho=0$, and that ν is bounded with bounded support. Let M be a continuous martingale with $\|M^*\|_p<\infty$, where $p^{-1}+\sum p_j^{-1}\le 1$, and assume that $N_J S_J$ is a.s. non-random even for $J=\{1,\dots,d\}$, unless M is constant. Then we have for any stopping time τ

$$E\, M_\infty \prod_{j=1}^d \int_{\tau+}^\infty V_j dX_j = E\, M_\tau \sum_\pi \prod_J N_J U_J(\tau), \tag{24}$$

where the summation extends over all partitions π of $\{1,\dots,d\}$ into sets J with $\#J\ge 2$.

Proof. We shall proceed by induction over d, starting for d=0 with the fact that M_∞ has optional projection M_t. Let us thus fix a $d\in N$, and assume that (24) is true with d replaced by $1,\dots,d-1$. To extend (24) to d, fix $T>0$, and proceed as follows, where each step will be explained *in detail below*:

$$E\,M_\infty \prod_{j=1}^{d} \int_{\tau+}^{\infty} V_j\,dX_j - E\,M_\infty \prod_{j=1}^{d} \int_{\tau\vee T+}^{\infty} V_j\,dX_j$$

$$= E\,M_\infty \sum_J \int_{\tau+}^{\tau\vee T+} V_J(t)\,dX_J(t) \prod_{j\notin J} \int_{t+}^{\infty} V_j\,dX_j \qquad \text{(integration by parts)}$$

$$= E \sum_J \int_{\tau+}^{\tau\vee T+} V_J(t)\,dX_J(t) M_t \sum_{\pi'} \prod_I N_I U_I(t) \qquad \text{(optional projection)}$$

$$= E \sum_J N_J \int_{\tau}^{\tau\vee T} V_J(t)\,dt\; M_t \sum_{\pi'} \prod_I N_I U_I(t) \qquad \text{(dual predictable projection)}$$

$$= E\,M_\infty \sum_J N_J \int_{\tau}^{\tau\vee T} V_J(t)\,dt \sum_{\pi'} \prod_I N_I U_I(t) \qquad \text{(optional projection)}$$

$$= E\,M_\infty \sum_\pi \prod_J N_J U_J(\tau) - E\,M_\infty \sum_\pi \prod_J N_J U_J(\tau\vee T) \qquad \text{(integration by parts)}$$

$$= E\,M_\tau \sum_\pi \prod_J N_J U_J(\tau) - E\,M_\infty \sum_\pi \prod_J N_J U_J(\tau\vee T) \qquad \text{(optional sampling).}$$

Here the integration by parts in the first step is according to Lemma 5.6 but in reversed time. The optional projection in the second step is by the induction hypothesis plus Lemma 5.7. Note that the innner summation on the right extends over all partitions π' of the set $\{1,\dots,d\}\setminus J$ into subsets I of size ≥ 2, and that the integrability requirements are fulfilled by Proposition 2.2 and Lemma 5.3. Since the new integrands are continuous adapted, and hence predictable, we may proceed in the next step by a dual predictable projection, where each process $X_J(t)$ is replaced by its compensator, which is $N_J t$ if $\#J\geq 2$ and vanishes otherwise. Note that the outer summation on the right is restricted to subsets $J\subset\{1,\dots,d\}$ with $\#J\geq 2$. The third step is formally justified by Proposition 2.2 with $p=1$, where the integrability condition follows from the fact that, by Hölder's inequality and Lemma 5.2,

$$E\int_0^T \left|V_J M \prod_I U_I\right| \leq \|M^*\|_p \left\|\int_0^T V_J\right\|_{p_J} \prod_I \|U_I^*\|_{p_I} < \infty. \tag{25}$$

In step number four we are using Lemma 5.7 again to replace M_t by M_∞, where the required integrability conditions now follow as in (25). Step number six is again by reversed integration by parts, as in Lemma 5.6. The sum in the first term is now $\mathcal{F}_\tau$-measurable, unless M is constant, so we may replace M_∞ by

$E[M_\infty|\mathcal{F}_\tau]=M_\tau$, which yields the final expression.

To complete the proof, it remains to notice that, by Hölder's inequality

$$\left|E\, M_\infty \prod_{j=1}^{d} \int_{\tau\vee T+}^{\infty} V_j dX_j\right| \leq \|M^*\|_p \prod_{j=1}^{d} \left\|\int_{\tau\vee T+}^{\infty} V_j dX_j\right\|_{p_j},$$

while

$$\left|E\, M_\infty \sum_\pi \prod_J N_J U_J(\tau\vee T)\right| \leq \|M^*\|_p \sum_\pi \prod_J |N_J| \left\|\int_{\tau\vee T}^{\infty} |V_J|\right\|_{p_J},$$

where the expressions on the right tend to zero as $T\to\infty$, by Proposition 2.2 and Lemma 5.2 plus dominated convergence. □

The following simple result will be needed to prove Theorem 5.1 when d=2.

Lemma 5.9. Fix $p,q\geq 1$ with $p^{-1}+q^{-1}\leq 1$, and let M and N be martingales with $\|M^*\|_p<\infty$ and $\|N^*\|_q<\infty$. Then

$$E\, M_\infty N_\infty = E\, M_0 N_0 + E[M,N]_\infty . \tag{26}$$

Proof. By the BDG and Hölder inequalities,

$$E\sup_t \left|\int_0^t M_- dN\right| \lesssim E\left\{\int_0^\infty M_-^2 d[N,N]\right\}^{1/2} \leq E\, M^*[N,N]_\infty^{1/2} \leq \|M^*\|_p \|[N,N]_\infty^{1/2}\|_q$$
$$\lesssim \|M^*\|_p \|N^*\|_q < \infty,$$

and similarly with M and N interchanged, so the processes $\int M_- dN$ and $\int N_- dM$ are uniformly integrable martingales, and (26) follows from Itô's formula. □

Proof of Theorem 5.1. Since the assertion holds for d=1 by Proposition 2.2, we may assume that $d\geq 2$. In that case the products $\gamma_j S_j$ are non-random, so we get with $X_j'(t)=X_j(t)-\gamma_j t$

$$E\prod_{j=1}^{d}\int_0^\infty V_j dX_j = \sum_J \prod_{j\in J}\gamma_j S_j\; E\prod_{j\notin J}\int_0^\infty V_j dX_j', \tag{27}$$

where the summation on the right extends over all subsets $J\subset\{1,\dots,d\}$. Thus we may henceforth assume that X is centered. By Lemma 5.8 we may further take ν to be bounded and boundedly supported.

Write B and Y for the continuous and purely discontinuous components of X, and denote the integral processes $\int V_j dB_j$ by M_j. Then the quantities

$$[M_i,M_j]_\infty = \int_0^\infty V_i V_j d[B_i,B_j] = \rho_{ij} S_{ij}, \quad i\neq j, \tag{28}$$

are non-random when $d\geq 3$, so in that case there exist by Lemma 4.2 some continuous

martingales M_J, $J\subset\{1,\ldots,d\}$, satisfying $M_J(\infty)=\prod_J M_j(\infty)$, $\|M_J^*\|_{p_J}<\infty$, and

$$E\,M_J=\sum_{\pi'}\prod_{i,j}\rho_{ij}S_{ij},\qquad \emptyset\neq J\subset\{1,\ldots,d\}, \tag{29}$$

where the summation on the right extends over all partitions π' of $\{1,\ldots,d\}$ into pairs $\{i,j\}$. Putting $M_J\equiv 1$ when $J=\emptyset$, we get by Lemma 5.8 with $\tau=0$

$$E\prod_{j=1}^{d}\int_0^\infty V_j\,dX_j = E\sum_J M_J(\infty)\prod_{j\notin J}\int_0^\infty V_j\,dY_j = E\sum_{\pi}\prod_{i,j}\rho_{ij}S_{ij}\prod_J N_J S_J, \tag{30}$$

with summations first over arbitrary subsets $J\subset\{1,\ldots,d\}$, and next over partitions π of $\{1,\ldots,d\}$ into pairs $\{i,j\}$ and subsets J with $\#J\geq 2$. This completes the proof for $d\geq 3$, so it remains only to take $d=2$. But in that case the first equality in (30) holds with $M_{12}=M_1M_2$, as does the second one, since by (28) and Lemma 5.9

$$E\,M_1(\infty)M_2(\infty)=E[M_1,M_2]=E\,\rho_{12}S_{12}.$$

□

6. Predictable transformations

We have seen already in Section 1 how the results and methods of this paper yield a unified approach to various invariance theorems in exchangeability. Here we shall continue this discussion by looking at some further examples. We shall further state our general reduction Theorem 6.4, and give a few applications. Most results in this section are either easy consequences of earlier statements or follow by similar arguments, so we shall be rather brief, and leave most details to the reader.

Let us first consider the case of separately exchangeable sequences or processes in R^d. By definition, these are invariant in distribution under possibly different permutations in the d components. We may further define the notion of separate $\mathcal{F}$-exchangeability, by requiring the above property to hold conditionally after time t, given $\mathcal{F}_t$. As before, there are unique de Finetti-type representations in terms of ergodic distributions, and it is easy to see that the latter correspond to sequences or processes where the d components are independent and ergodic in the usual sense.

Proposition 6.1. Fix $I=\{1,\ldots,n\}$ or N, and consider random variables ξ_{jk} and τ_{jk}, $j=1,\ldots,d$, $k\in I$, such that $\xi=(\xi_{jk})$ is separately $\mathcal{F}$-exchangeable, while the τ_{jk} are $\mathcal{F}$-predictable stopping times in I and a.s. distinct for fixed j. Then $(\xi_{j,\tau_{jk}}) \stackrel{d}{=} \xi$. If instead X and U are processes on $I=[0,1]$ or R_+, such that X is R^d-valued and separately $\mathcal{F}$-exchangeable while U is T^d-valued and $\mathcal{F}$-predictable with $\lambda U_j^{-1}=\lambda$ a.s. for each j, then $(X_1U_1^{-1},\ldots,X_dU_d^{-1}) \stackrel{d}{=} X$.

It is not hard to extend the one-dimensional arguments of [12] to the present context. An interesting alternative is to use decoupling identities, as follows. Consider e.g. the continuous time case, and assume for simplicity that X is ergodic. If the time scale is R_+, we may reduce to the case of bounded jumps by a simple truncation. Now fix simple step functions $f_1,\ldots,f_d$ with compact support, and note that

$$\int (f_j \circ U_j)^m d\lambda = \int f_j^m d(\, U_j^{-1}) = \int f_j^m d\lambda, \qquad j=1,\dots,d,\ m\in N.$$

By Theorem 4.1 or 5.1 we get, for any $m_1,\dots,m_d \in Z_+$,

$$E \prod_{j=1}^d \left\{ \int f_j\, d(X_j U_j^{-1}) \right\}^{m_j} = E \prod_{j=1}^d \left\{ \int f_j \circ U_j\, dX_j \right\}^{m_j} = E \prod_{j=1}^d \left\{ \int f_j\, dX_j \right\}^{m_j}.$$

Since the moments determine the distribution in this case, by Proposition 2.2 or 2.8, we obtain

$$\sum_{j=1}^d \int f_j\, d(X_j U_j^{-1}) \overset{d}{=} \sum_{j=1}^d \int f_j\, dX_j,$$

and the assertion follows by the Cramér-Wold theorem.

A similar argument applies to infinite sequences. For finite sequences, however, ergodicity in the sense of separate exchangeability does not imply ergodicity in the joint sense. For this reason, one needs the following extended version of Theorem 3.1, which may be proved in the same way.

Lemma 6.2. *Let ξ and η be random n-sequences in R^d, such that ξ is jointly $\mathcal{F}$-exchangeable while η is $\mathcal{F}$-predictable. Suppose that the partition $\{K_1,\dots,K_m\}$ of $\{1,\dots,d\}$ splits ξ into independent ergodic sequences. Let (C_1) be fulfilled, and assume that the sums S_J with $J\subset K_i$ for some i are a.s. non-random. Then*

$$E \prod_{j=1}^d \sum_{k=1}^n \xi_{jk}\eta_{jk} = \prod_{i=1}^m P_{n,K_i}\{R_J, S_J\}, \tag{1}$$

for some polynomials P_{n,K_i} in the sums R_J and S_J with $J\subset K_i$.

Larger classes of predictable transformations preserve the distributions of *continuous* exchangeable processes.

Proposition 6.3. *Let B be an $\mathcal{F}$-exchangeable Brownian motion on $I=R_+$ or bridge on $I=[0,1]$, and let V_t, $t\in I$, be a family of $\mathcal{F}$-predictable processes on I with $\int V_t(u)du=0$ a.s. for each t, and such that respectively*

$$\int_I V_s(u)V_t(u)du = s\wedge t \quad\text{or}\quad s\wedge t - st \quad \text{a.s.}, \qquad s,t\in I. \tag{2}$$

Then B has the same distribution as the process

$$Y_t = \int_I V_t(u)\, dB_u, \qquad t\in I. \tag{3}$$

Proof. We may e.g. take $I=[0,1]$. By Theorem 4.1 we get for any $d\in N$ and

$t_1,\ldots,t_d \in I$

$$E\prod_{j=1}^{d} Y_{t_j} = \sum_{\pi}\prod_{i,j}(t_i \wedge t_j - t_i t_j) = E\prod_{j=1}^{d} B_{t_j},$$

where π denotes an arbitrary partition of the set $\{1,\ldots,d\}$ into pairs $\{i,j\}$. Since Gaussian distributions are characterized by their moments, it follows that $Y \overset{d}{=} B$. $\square$

An interesting example is to take $V=(V_t)$ to be an ergodic exchangeable process on I, defined on the Lebesgue unit interval as a probability space. Assuming the directing triple (α,σ^2,β) or (γ,σ^2,ν) to be such that $\alpha=0$ and $\sigma^2+\sum\beta_j^2=1$, or $\gamma=0$ and $\sigma^2+\int x^2\nu(dx)=1$, respectively, it is easily seen that V satisfies the conditions of Proposition 6.3, with the integrals taken over $[0,1]$. If $I=R_+$, we may extend the definition to I by putting $V_t(u)=0$ for $u>1$. Note that the functions V_t, regarded as processes on I, are non-random in this case and hence trivially predictable. Intuitively, the process Y is obtained here as a 'stochastic average' over the paths of V.

Our next aim is to state our general reduction theorem mentioned earlier. Though this result is not directly related to exchangeability, we include it here since the proof is essentially the same as for Theorem 5.1. The statement requires some terminology.

Given a Polish space K, we define a point process in K to be a locally finite and integer valued random measure ξ on $R_+\times K$, such that $\xi(\{t\}\times K)\le 1$ for all t. Fix a standard filtration $\mathcal{F}$ indexed by R_+, and say that ξ is adapted to $\mathcal{F}$, if the process $N_t(B)=\xi([0,t]\times B)$ is adapted for every bounded Borel set $B\subset K$. In this case the associated compensator $\hat{N}_t(B)$ has a measure valued version, and we may define the compensator of ξ to be the a.s. unique random measure $\hat{\xi}$ on $R_+\times K$ satisfying $\hat{\xi}([0,t]\times B)=\hat{N}_t(B)$ for all t and B. Say that $\hat{\xi}$ is continuous if $\hat{N}_t(B)$ is a.s. continuous for every B as above. Given an arbitrary space S, we may form S_∂ by adding an extraneous state ∂. Finally recall that a covariance function on an abstract space T is a non-negative definite function $\rho: T^2\to R$.

Theorem 6.4. Fix a Polish space K, *a* σ-*finite measure space* (S,μ), *and a covariance function* ρ *on some space* T. *Let* ξ *be an adapted point process in* K *with continuous compensator* $\hat{\xi}$, *and let* $M_1,\ldots,M_d$ *be continuous local martingales. Consider predictable processes* $V: \Omega\times R_+\times K \to S$, *and* $U_{1t},\ldots,U_{dt}: \Omega\times R_+ \to R$, $t\in T$, *with* $\sum_j \int U_{jt}^2 d[M_j,M_j] < \infty$ *a.s. for all* t, *and such that*

$$\hat{\xi}V^{-1} = \mu \quad \text{a.s.}; \qquad \sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^\infty U_{is}U_{jt}\,d[M_i,M_j] = \rho_{st} \quad \text{a.s.}, \qquad s,t\in T. \tag{5}$$

Define a random measure η *on* S *and a random field* Y *on* T *by putting*

$$\eta = \xi V^{-1}; \qquad Y_t = \sum_{j=1}^{d}\int_0^\infty U_{jt}\,dM_j, \qquad t\in T. \tag{6}$$

Then η *and* Y *are independent, and* η *is Poisson with* $E\eta=\mu$, *while* Y *is centered Gaussian with* $EY_sY_t=\rho_{st}$.

The proof depends on the following lemma, which may be obtained in the same way as Lemma 5.8.

Lemma 6.5. Let ξ, $\hat{\xi}$ *and* V *be such as in Theorem 6.4, and let* M *be a continuous local martingale with bounded quadratic variation. Fix a stopping time* τ *and measurable sets* $B_j\subset S$ *with* $\mu B_j<\infty$, $j=1,\ldots,d$. *Then*

$$E\,M_\infty \prod_{j=1}^{d}\xi\{(t,x);\ t>\tau,\ V_{t,x}\in B_j\} = E\,M_\tau \sum_{\pi}\prod_{J}\hat{\xi}\{(t,x);\ t>\tau,\ V_{t,x}\in \bigcap_{j\in J} B_j\}, \tag{7}$$

where π *denotes an arbitrary partition of* $\{1,\ldots,d\}$ *into subsets* J.

Proof of Theorem 6.4 (outlined). Fix $m,n\in Z_+$, $t_1,\ldots,t_m\in T$, and $B_j\subset S$ measurable with $\lambda B_j<\infty$, $j=1,\ldots,n$. By Lemmas 4.2 and 6.5,

$$E\prod_{i=1}^{m}Y_{t_i}\prod_{j=1}^{n}\eta B_j = \sum_{\pi_1}\prod_{i,j}\rho(t_i,t_j)\sum_{\pi_2}\prod_{J}\mu\bigcap_{j\in J}B_j, \tag{8}$$

where π_1 and π_2 are arbitrary partitions of the sets $\{1,\ldots,m\}$ and $\{1,\ldots,n\}$ into pairs $\{i,j\}$ and subsets J, respectively. To see that this is the desired moment, consider the special case when $(M_1,\ldots,M_m)$ is a Brownian motion in R^m with covariances $\rho(t_i,t_j)$ at time 1, while ξ is an independent Poisson process on R_+ with unit intensity, and choose non-random $U_1,\ldots,U_m$ and V such that $Y(t_j)=M_j(1)$ and $\lambda V^{-1}=\mu$. $\square$

We conclude with some simple illustrations. Let us first take $M_1,\dots,M_d$ orthogonal with $M_j(0)=0$ and $[M_j,M_j]_\infty=\infty$. Let $T=\{1,\dots,d\}\times R_+$, and define

$$U_{ij}(s,t) = \delta_{ij} 1\{[M_i,M_j]_s<t\}, \quad i,j=1,\dots,d, \quad t\geq 0. \tag{9}$$

Writing $\tau_j(t)=\inf\{s;\ [M_j,M_j]_s=t\}$, it is seen from Theorem 6.4 that

$$Y_j(t) = \int 1\{[M_j,M_j]_s<t\}dM_j(s) = M_j\circ\tau_j(t), \quad j=1,\dots,d, \quad t\geq 0, \tag{10}$$

are independent Brownian motions, as shown by Knight (1970) (though our theorem gives no information about paths).

Next we take $K=\{1,\dots,d\}$, write $\xi=(\xi_1,\dots,\xi_d)$ and $\hat\xi=(\hat\xi_1,\dots,\hat\xi_d)$, and assume that $\hat\xi_j(\infty)=\infty$ for all j. (Here and below, $\xi_j(t)=\xi_j[0,t]$, etc.) Choose $S=\{1,\dots,d\}\times R_+$ with Lebesgue measure μ, and define

$$V_j(t) = (j,\ \hat\xi_j(t)), \quad j=1,\dots,d, \quad t\geq 0. \tag{11}$$

Putting $\tau_j(t)=\inf\{s;\ \hat\xi_j(s)=t\}$, our theorem shows that

$$\eta_j(t) = \xi_j\{s;\ \hat\xi_j(s)\leq t\} = \xi_j\circ\tau_j(t+), \quad j=1,\dots,d, \quad t\geq 0, \tag{12}$$

are independent Poisson processes with rate 1, as noted by Meyer (1971). (The processes $\xi_j\circ\tau_j(t)$ will clearly have the same property.)

For a less trivial example, let X be a strictly p-stable motion (Lévy process) on R_+, and assume for simplicity that $p<1$, so that $X_t=\sum_{s\leq t}\Delta X_s$, $t\geq 0$. Recall that X has Lévy measure $\nu=a\nu_p^+ + b\nu_p^-$ for some constants $a,b\geq 0$, where $\nu_p^\pm$ denote the measures on $(0,\infty)$ and $(-\infty,0)$ with density $|x|^{-1-p}$. Define a point process ξ in $R\setminus\{0\}$ by

$$\xi_t(B) = \sum_{s\leq t} 1\{\Delta X_s\in B\}, \quad t\geq 0, \quad B\subset R\setminus\{0\}, \tag{13}$$

so that ξ is Poisson with intensity=compensator $\hat\xi=\lambda\times\nu$.

Let us now fix a predictable process U on R_+ with $\int_0^t |U|^p<\infty$ a.s. for all t. Write $U=U^+-U^-$ where $U^+=U\vee 0$, and define

$$T_t^\pm = \int_0^t (U_s^\pm)^p ds, \quad V_{t,x}^\pm = (T_t^\pm,\ xU_t^\pm), \quad t\geq 0,\ x\in R\setminus\{0\}. \tag{14}$$

If we assume $T_\infty^\pm=\infty$ a.s., it is easily seen that $(\lambda\times\nu)(V^\pm)^{-1}=\lambda\times\nu$, so Theorem 6.4 shows that the measures $\xi^\pm=\xi(V^\pm)^{-1}$ are mutually independent copies of ξ. Hence the processes

$$X_t^{\pm} = \int x\, \xi^{\pm}([0,t]\times dx) = \int U_s^{\pm} 1\{T_s^{\pm} \leq t\} dX_s, \qquad t \geq 0, \tag{15}$$

are mutually independent copies of X. Thus the process $\int UdX$ exists (which is known from [9]) and has a.s. representation

$$\int_0^{t+} UdX = \int U_s^{+} 1\{T_s^{+} \leq T_t^{+}\} dX_s - \int U_s^{-} 1\{T_s^{-} \leq T_s^{-}\} dX_s = X^{+}\circ T_t^{+} - X^{-}\circ T_t^{-}, \qquad t \geq 0. \tag{16}$$

A similar argument yields the alternative a.s. representation

$$\int_0^{t+} UdX = Y^{+}(aT_t^{+} + bT_t^{-}) - Y^{-}(aT_t^{-} + bT_t^{+}), \qquad t \geq 0, \tag{17}$$

for some mutually independent stable motions Y^{+} and Y^{-} with Lévy measure ν_p^{+}. For a=b we recover the time change result of Rosiński & Woyczyński (1986). (An alternative approach is via Proposition 6.1 above. The case when $P\{T_\infty^{\pm} < \infty\} > 0$ is similar but requires a randomization, e.g. as in [12] or in [4], pp. 89ff.)

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